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ABSTRACT

One way to stimulate interest in mathematics among students is to provide them with enrichment activities which often are somewhat different than regular classroom activities. Based on that premise, this publication presents a compendium of activities designed to enrich mathematics at various grade levels. The activities fall into seven categories. Three groups are simply categorized by grade level; namely primary, intermediate, and middle school. The algebra I activities are designed to be treated at whichever grade that subject is taught, while the activities designated for higher algebra are intended for students studying algebra beyond the first year. The geometry level is designed for students enrolled in a traditional high school geometry course, and the advanced mathematics category contains topics usually compatible with students taking a senior level mathematics course. Each activity in the document includes an introduction, objectives, a list of materials needed, and a description of the procedure. Some activities also contain reproducible worksheets suitable for use by the students. Solutions to problems are also provided for the teacher. (TW)

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MATHEMATICS ENRICHMENT PROGRAM

GRADES 3-12

Commonwealth of Virginia
Department of Education
Division of Sciences and
Elementary Administration
Mathematics Service
Richmond, Virginia 23216
May 1986

SE 048 267

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MATHEMATICS ENRICHMENT ACTIVITIES
GRADES 3 THROUGH 12

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PREFACE

One of the most effective ways of stimulating interest in mathematics among students is to provide them with enrichment activities which oftentimes are somewhat different from regular classroom activities. Although it is sometimes argued that such digressions consume valuable time which is needed to complete the course syllabus, the enhanced attitude towards mathematics among your students resulting from such enrichment activities will more than offset this 'loss' of time ordinarily spent on the topics prescribed in the course syllabus. In that spirit, this publication presents a compendium of activities designed to enrich mathematics instruction at the various grade levels.

Enrichment in mathematics is not reserved for gifted students. Hence, this book offers a set of enrichment activities to be used with classes at various ability levels. In some cases the same activity may be used with both a class of highly capable students and an average ability class. The teacher is required to make the proper adjustments to gear the activity appropriately. In essence this allows the teacher to take an activity from a section designed for use with one type of student and use it with an entirely different audience. The categorization of activities was done according to the most appropriate placement of the particular activity.

The activities fall into seven groups or categories. Three groups are simply categorized by grade level; the Algebra I activities may be treated at whichever grade that subject is studied, while the activities designated Higher Algebra is intended for students studying algebra beyond the first year course. The Geometry level is designed for students enrolled in a traditional high school geometry course, and the Advanced Mathematics category contains topics usually compatible with students taking a senior level mathematics course in high school.

The Enrichment Activities have been designed for use by teachers in planning for such activities with their classes. It is not expected that teachers will use the activities exactly as presented. Teachers are expected to tailor these activities to the particular audience for which they are intended. Some of the activities may serve as sources for independent study, while others may be best used as the topic for class consideration during one or more class recitations. Although guidelines for presenting these activities are offered throughout, the final judgement for style of presentation rests with the individual teacher.

ACKNOWLEDGEMENTS

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Many sources were used in the development of these activities. The major source was *Teaching Secondary School Mathematics: Techniques and Enrichment Units,* by Alfred S. Posamentier and Jay Stepelman: Charles E. Merrill Publishing Company, Second Edition, 1986.

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TITLES BY GRADE LEVEL AND COURSE

The activities listed below are grouped by grade level and/or according to a branch of Mathematics. If an activity is designed for use in one or more categories the activity is cross referenced in the lower grade level and listed in other appropriate categories.

KEY

A.	K–3
B.	4–6
C	7–8
D	ALGEBRA I
E.	ALGEBRA II
F.	GEOMETRY
G	ADVANCED MATHEMATICS

GRADE LEVEL	TITLE
A	DEVELOPING A PROBLEM SOLVING STRATEGY
A	EVEN STEVEN ODDS AND EVEN
A	WHAT'S IN A NUMBER
A, B	WHAT'S MISSING (PATTERNS)
B	AUSTRIAN SUBTRACTION
B	CASTING OUT NINES
B	DRAWING WITH RECTANGULAR COORDINATES
B	EGYPTIAN ARITHMETIC G RULE
B	EXPERIMENTS IN SYMMETRY
B	FOUR FOURS
B	HAPPY NUMBERS
B	LATTICE MULTIPLICATION
B	MAGIC SQUARE
B	MEASUREMENT AND CONSTRUCTION
B, C	NUMBER ODDITIES
B, C	PATTERNS IN REPEATING DECIMALS
B	PROBLEM SOLVING STRATEGIES
B	PYTHAGOREAN THEOREM
B	SHORT CUT ALGORITHMS FOR MULTIPLES OF 5
B	A 6-SIEVE FOR PRIMES
B	SQUARING A NUMBER ENDING IN 5
B	SUM OF ANGLE MEASURES OF POLYGONS
B, C, F	TOPOLOGY EQUIVALENCE
B	TRANSFORMATIONS WITH ORDERED PAIRS
B	WHAT'S MISSING
C	BELIEVING THAT IT CAN BE SOLVED
C	CHANGING REPEATING DECIMALS TO FRACTIONS
C, F	CURVE STITCHING
C	FITTING A CIRCLE INTO A PARALLELOGRAM
C	GRAPHING AND SPATIAL RELATIONS
C, D	GREATEST COMMON FACTOR OF MORE THAN TWO NUMBERS
C, F	HEXAGONAL FLIPS
C	INVESTIGATING SQUARES AND CUBES
C, D, E, F, G	MÖBIUS STRIP
C	NUMBER ODDITIES
C, D	PASCAL'S TRIANGLE
C	PATTERNS IN REPEATING DECIMALS
C	PROBABILITY AND PASCAL'S TRIANGLE

C, F	PYTHAGOREAN THEOREM
C	SEQUENCES USING TABLES OF DIFFERENCES
C	SHORT CUTS IN ARITHMETIC
C	SUM OF N ODD NUMBERS
C	SYMMETRY OPERATIONS
C	TOPOLOGY EQUIVALENCE
C	TRIANGULAR NUMBERS
D	EXPONENTS AND RADICALS
D	FACTORIZATION OF POLYNOMIALS RELATED TO AREA FUNCTIONS
D	GREATEST COMMON FACTOR OF MORE THAN TWO NUMBERS
D	MAGIC TRIANGLES
D	MATH TRICKS
D	MIND READING TRICKS
D	MÖBIUS STRIP
D	MODULO ARITHMETIC
D	PERFECT NUMBERS
D, E	POMPEII'S MAGIC SEVEN
E, F	A PIECE OF PI
E, G	CONIC ART
E	FALLACIES IN PROOFS
E, F	FOLDING PAPER CONIC SECTIONS
E, G	GROUPS
E	MAGIC IN THREE-DIGIT NUMBERS
E	MÖBIUS STRIPS
E, G	NUMBER THEORY PROBLEMS INVOLVING CONGRUENCE AND DIVISIBILITY
E	POMPEII'S MAGIC SEVEN
E	SEQUENCES AND SERIES
E	UNFINISHED BUSINESS OF MATHEMATICS
F	A PIECE OF PI
F	ARBELOS
F, G	BASEBALL PROBABILITY DIAMOND
F	BISECTING A LINE SEGMENT
F	CONSTRUCTING A REGULAR 17 — GON
F	CONSTRUCTION OF A SEGMENT OF LENGTH $\sqrt{X}$
F	CONSTRUCTIONS AND DISSECTIONS
F	CURVE STITCHING
F	FOLDING PAPER CONIC SECTIONS
F	HEXAGONAL FLIP
F	MÖBIUS STRIP
F	NETWORKS
F	PYTHAGOREAN THEOREM
F	REULEAUX TRIANGLES AND OTHER CURVES OF CONSTANT WIDTH
F	SQUARES INSCRIBED IN RIGHT TRIANGLES
F	STEWART'S THEOREM
F	TAXI-CAB DISTANCES
F	TOPOLOGY EQUIVALENCE
G	BASEBALL PROBABILITY DIAMOND
G	CHALLENGING COUNTING PROBLEMS
G	CHANCES OF WINNING A LOTTERY WITH A BUCK
G	CONIC ART
G	FAREY SQUARES
G	GROUPS
G	MÖBIUS STRIP
G	NUMBER THEORY PROBLEMS INVOLVING CONGRUENCE AND DIVISIBILITY

A PIECE OF PI

GRADE LEVEL/ COURSE: Algebra II, Geometry

INTRODUCTION/ OBJECTIVES: This topic will be interesting to those who enjoy history and entertaining to those who are trivia buffs. It is an example of the study of math for its "aesthetic value". Students should be familiar with the concept of Pi and its relationship to a circle.

1. Students will demonstrate a knowledge of the history of Pi.
2. Students will recite the value of Pi to varying decimal places, ranging from 5 to 30.
3. Students will examine Archimedes' classical method for approximating the value of Pi.

HISTORY: Most students recognize 3.14 as the value of Pi (π). Many people down through time have attempted to establish a value for Pi. For example, the General Assembly of Indiana in 1897 enacted a law to declare Pi legally equal to 4 (This is the most inaccurate version ever used). In the same year the Indiana State Legislature came within a single vote of declaring Pi equal to 3.2.

Pi has a long history. The ancient Egyptians discovered the nature of Pi. (This is true although π is a Greek letter.) The Egyptians calculated the value of Pi to be $\left(\frac{4}{3}\right)^4$ $\left(\frac{4}{3}\right)^4 = 256/81$ or $3.16^+ \ldots$, which is quite accurate.

The fact that Pi is irrational and is therefore a nonterminating, nonrepeating decimal was shown by the German mathematician Lambert in 1761.

Archimedes, the famous Greek mathematician, found that Pi was between 3 1/7 and 3 10/71. The average of these two values is 3.141585 . . . , differing from Pi only in the ten-thousandth place!

The first actual formula that could be used to calculate Pi was discovered by Viete of France:

$$2/\pi = \sqrt{1/2} \times \sqrt{1/2 + 1/2\sqrt{1/2}} \times \sqrt{1/2 + 1/2\sqrt{1/2 + 1/2\sqrt{1/2}}} \ldots$$

James Gregory of Scotland developed this formula:

$$4/\pi = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ldots}}}}}$$

Leibniz's Formula for Pi was $\pi/2 = \dfrac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \ldots}{3 \times 3 \times 5 \times 5 \times 7 \times \ldots}$

Another formula is $\pi^2/6 = 1 + (1/2)^2 + (1/3)^2 + (1/4)^2 + (1/5)^2 \ldots$

The students will note that these formulas can be evaluated easily using today's calculators and computers. They should be reminded, however, that before our electronic age mathematicians evaluated these formulas laboriously by hand, searching for signs of repetition. In some German texts Pi is frequently called "Ludolf's number", in honor of the 15th century mathematician who determined the value of Pi to 35 places.

It was estimated in about 1904 that it would require approximately 10 years of computation to find the value of Pi to 1000 decimal places.

William Shanks spent 15 years calculating Pi to 707 places. His record stood until 1949 when a computer called ENIAC, after 70 hours of computation spewed out some 2000 digits of Pi. In the early 1960's an electronic computer evaluated Pi to more than 100,000 decimal places in less than 9 hours. More recently, a modern computer calculated Pi to 1,000,000 places.

Besides computing the value of Pi, memorizing its value has been a favorite pastime of mathematicians. One British mathematician memorized the number to 5050 places. Various phrases or bits of prose, called mnemonics, in which the number of letters in successive words gives the successive digits of π have been devised to remind one of its value. "$\overset{3}{\text{How}}$ $\overset{1}{\text{I}}$ $\overset{4}{\text{wish}}$ $\overset{1}{\text{I}}$ $\overset{5}{\text{could}}$ $\overset{9}{\text{recollect}}$ $\overset{2}{\text{Pi}}$ $\overset{6}{\text{easily}}$ $\overset{5}{\text{today}}$" gives Pi to 8 decimal places, and "May I have a large container of coffee" to 7. "Now I live a dreary existence in ragged suits and cruel taxation suffering" gives 12 digits of Pi.

An easy way to remember Pi to 5 places is the simple device that 1 × 4 + 1 × 5 = 9.

This poem won a mnemonic contest sponsored by an English banking magazine. It specifies Pi to 30 digits.

Now I will a rhyme construct	3.14159
By chosen words the young instruct,	265358
Cunningly devised endeavour,	979
Con it and remember ever,	32384
Widths in circles here you see,	626433
Sketched out in strange obscurity.	83279.

If any of your students are curious, here are the next 10 places of Pi: 50288 41971.

Archimedes' Classical Method for Approximating the Value of π

ACTIVITY: Begin with a unit circle; that is, a circle with radius equal 1. The circumference of this circle is

$$C = 2\pi r = 2\pi(1) = 2\pi$$

Therefore,

$$\pi = \frac{C}{2}$$

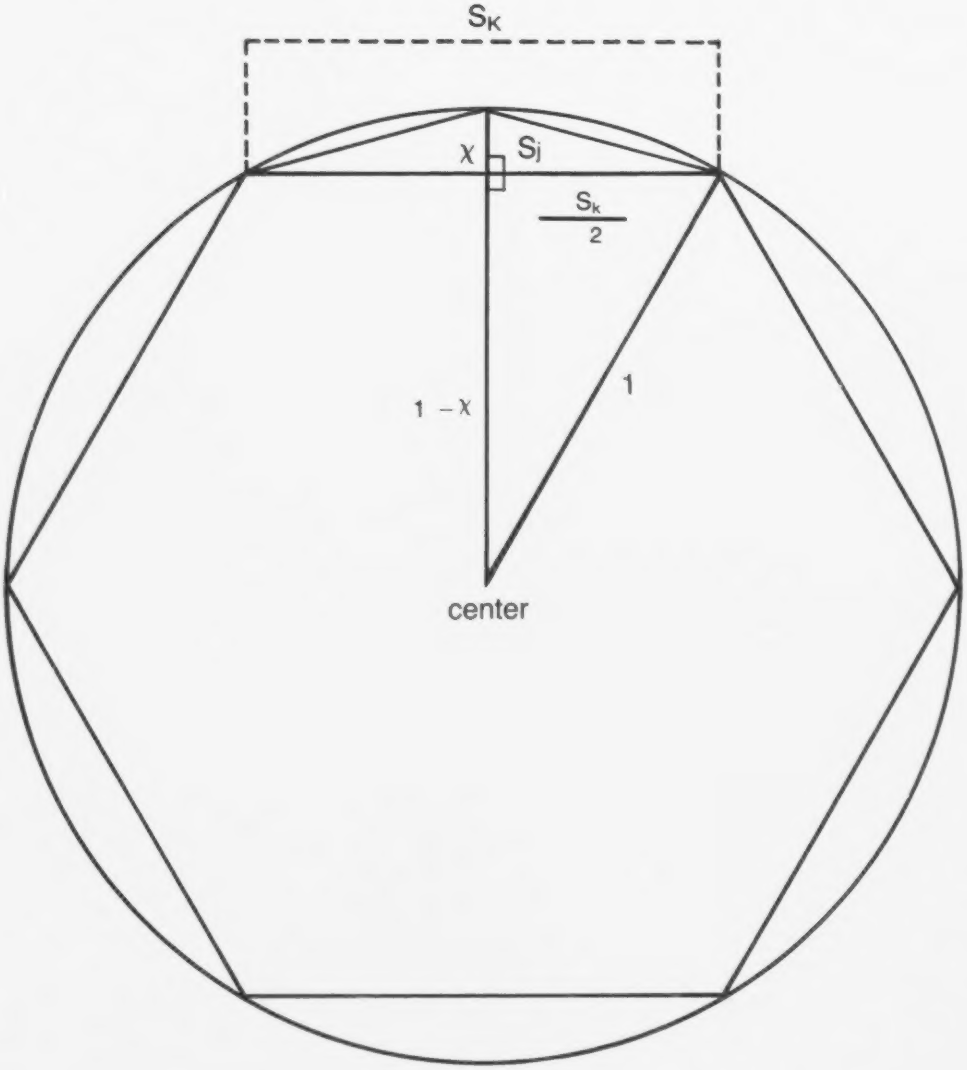
Next, inscribe a regular polygon of k sides, and let s_k denote the length of any side. The perimeter, P, of this polygon will be

$$P = k(s_k)$$

The perimeter of this polygon will be close in value to the circumference of the circle if the number of sides of the polygon is very large. Therefore,

$$\text{(I)} \qquad \pi = \frac{C}{2} \doteq \frac{P}{2} = \frac{k(S_k)}{2} \text{ if k is very large.}$$

We can improve the accuracy of our estimate of π by doubling the number of sides of the polygon. By continuing the doubling process the perimeter of the polygon will approach the circumference of the circle and a better approximate value of π can be obtained. Therefore, we first develop a formula to find the length of the side of a polygon with twice the number of sides as the original polygon.
Let s_j = length of side of polygon after the number of sides has been doubled.



Referring to the figure and using the Pythagorean Theorem with first the small and then the large right triangles we develop formulas II and III which together gives us the desired formula IV.

Equation 1: $(s_j)^2 = x^2 + \left(\frac{s_k}{2}\right)^2$

Solving for x: $(s_j)^2 = x^2 + \left(\frac{s_k}{4}\right)^2$

(II) $x = \frac{\sqrt{4(s_j)^2 - (s_k)^2}}{2}$

Equation 2: $1^2 = (1-x)^2 + \left(\frac{s_k}{2}\right)^2$

$$1 - \left(\frac{s_k}{2}\right)^2 = (1-x)^2$$

$$\frac{4-(s_k)^2}{4} = (1-x)^2$$

$$\frac{\sqrt{4-(s_k)^2}}{2} = 1-x$$

(III) $$x = \frac{2-\sqrt{4-(s_k)^2}}{2}$$

Setting II = III we obtain:

$$\frac{\sqrt{4(s_j)^2-(s_k)^2}}{2} = \frac{2-\sqrt{4-(s_k)^2}}{2}$$

$$\sqrt{4(s_j)^2-(s_k)^2} = 2-\sqrt{4-(s_k)^2}$$

$$4(s_j)^2-(s_k)^2 = 4-4\sqrt{4-(s_k)^2}+4-(s_k)^2$$

$$(s_j)^2 = 2-\sqrt{4-(s_k)^2}$$

(IV) $$\therefore \boxed{s_j = \sqrt{2-\sqrt{4-(s_k)^2}}}$$

Now we can proceed to approximate the value of π. By inscribing a regular hexagon in our unit circle, we have a polygon all of whose sides equal the radius of the circle, i.e. $k = 6$, $s_k = s_6 = 1$, since the radius of the circle is 1.

$$\pi \doteq \frac{k(s_k)}{2} = \frac{(6)\,(1)}{2} \doteq 3$$

Doubling the number of sides to 12 and using formula IV, $S_j = S_{12}$ we get

$$S_{12} = \sqrt{2-\sqrt{4-(S_6)^2}}$$

$$S_{12} = \sqrt{2-\sqrt{4-1}}$$

$$S_{12} = \sqrt{2-\sqrt{3}}$$

$$S_{12} \doteq 51763809$$

Then using this value in formula I, we get

$$\pi \doteq \frac{12(S_{12})}{2} \doteq \frac{6.2116571}{2} \doteq 3.1058285$$

To obtain a closer approximation of π one needs to double the number of sides to 24 and use formulas IV and I.

$$S_{24} = \sqrt{2-\sqrt{4-(S_{12})^2}}$$

$$S_{24} = \sqrt{2-\sqrt{4-(.51763809)^2}}$$

$$S_{24} = .26105239$$

$$\pi \doteq \frac{(24)(S_{24})}{2} \doteq \frac{(24)(.26105239)}{2} \doteq 3.1326286$$

Note: Use of hand-held calculators that round off the last digit will introduce errors in the calculation of π.

EXTENDED LEARNING:

1. Advanced Geometry students could be asked to prove that $S_j = \sqrt{2 - \sqrt{4 - (S_k)^2}}$ is true.
2. For fun, have students make up their own mnemonic for remembering the value of Pi.

REFERENCES:

Britton, Jack R. and Bello, Ignacio. *Topics in Contemporary Mathematics.* New York: Harper and Row Publishers, 1984.

Lewis, David B. *Eureka! Math Fun from Many Angles.* New York: Putnam Publishing Group, 1983.

A 6-SIEVE FOR PRIMES
(MOVE OVER ERATOSTHENES!)

GRADE LEVEL: 4–6

INTRODUCTION/ OBJECTIVES: Students will use a 6-sieve to discover some interesting patterns and to formulate a conjecture about prime numbers. The classic method for finding primes is using the sieve of Eratosthenes.

MATERIALS: Colored pencils or pens, prepared sieves (optional)

PROCEDURE: The sieve can be used with a whole class or as an individual assignment with written directions. An answer key follows the activity.

ACTIVITY:

Step 1. Have students write numbers from 1 to 100 in horizontal rows of 6 (or hand them prepared sieves)

Step 2. Mark out 1 since 1 is not a prime number. Then use a colored pencil to circle 2 and mark out all the even numbers greater than 2.

Step 3. Use a different color pencil to circle 3 and mark out all the multiples of 3.

Step 4. Now describe the patterns of the marked out 2's and 3's (verbally or written)

Step 5. Circle 5 and mark out multiples of 5 using another color. Describe the pattern.

Step 6. Repeat this procedure for 7 and multiples of 7.

Step 7. Circle 11 and mark out multiples of 11. Do the multiples of 11 form a pattern?

Step 8. Circle the unmarked numbers. These are the primes.

Step 9. Have students list vertically the multiples of six between 6 and 96 with a blank on either side. The students can use the sieve to find the primes less than 100. If there is a prime number less than the number, write it in the first blank. If there is a prime number greater than the number, write it in the second blank. If there is no prime, do not write anything in the blank.

Answer to Step 9

(5) 6 (7)
(11) 12 (13)
(17 18 (19)
(23) 24 ___
(29) 30 (31)
___ 36 (37)
(41) 42 (43)
(47) 48 ___
(53) 54 ___
(59) 60 (61)
___ 66 (67)
(71) 72 (73)
___ 78 (79)
(83) 84 ___
(89) 90 ___
___ 96 (97)

Step 10. Is there any pattern or relationship which you can find for the prime numbers?

Step 11. Are there any even prime numbers? How do you know?

Step 12. Are all odd numbers prime? How do you know?

Step 13. Can you use the sieve to find the factors of 84? How?

	1	2	3	4	5	6
	7	8	9	10	11	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF TWO

	1	②	3	4	5	6
	7	8	9	10	11	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF THREE

	1	2	③	4	5	6
	7	8	9	10	11	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF FIVE

	1	2	3	4	⑤	6
	7	8	9	10	11	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF SEVEN

	1	2	3	4	5	6
	⑦	8	9	10	11	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF ELEVEN

	1	2	3	4	5	6
	7	8	9	10	⑪	12
	13	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

MULTIPLES OF THIRTEEN

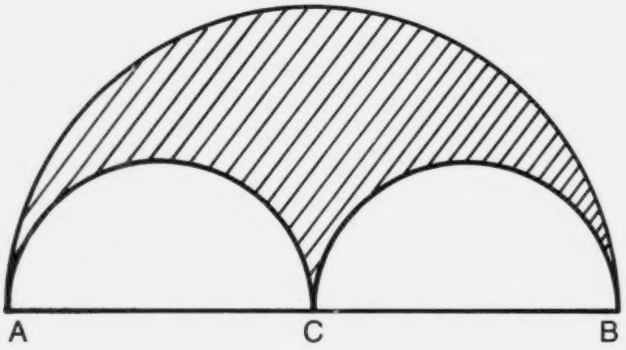
	1	2	3	4	5	6
	7	8	9	10	11	12
	⑬	14	15	16	17	18
	19	20	21	22	23	24
	25	26	27	28	29	30
	31	32	33	34	35	36
	37	38	39	40	41	42
	43	44	45	46	47	48
	49	50	51	52	53	54
	55	56	57	58	59	60
	61	62	63	64	65	66
	67	68	69	70	71	72
	73	74	75	76	77	78
	79	80	81	82	83	84
	85	86	87	88	89	90
	91	92	93	94	95	96
	97	98	99	100		

THE ARBELOS (AR-BEĹ-OS)

GRADE LEVEL/ COURSE: GEOMETRY

INTRODUCTION/ OBJECTIVES: The student will investigate the relationships found in the arbelos.

HISTORY: The "Arbelos" or the "Shoemaker's Knife" is a simple geometric figure studied by many mathematicians, including Archimedes (287–212 B.C.) in his *Book of Lemmas (Liber assumptorum).*



In the figure, let A, C, B be three collinear points such that C is between A and B. Semicircles are drawn on the same side of the line and have AC, CB, AB as diameters. The region (shared area) bound by these three semicircles is called the "Arbelos" and has many properties.

PROCEDURE:

1. The teacher should read and attempt the following worksheet before allowing either a student or a group of students to participate. A review of the following skills might be needed: arc lengths, areas of circles, properties of triangles, rectangles, and circles.
2. Worksheet answers and explanations follow the worksheet.

ACTIVITY: See worksheet.

Worksheet: The Arbelos

The arbelos is the shaded region within the semicircle. For this figure, x is any point between A and B where the radii of semicircles on $\overline{AB}$, $\overline{AX}$, $\overline{XB}$ are R, r_1, r_2.

Step 1. Determine the following arclengths:

$m\,\overset{\frown}{AB}$ = ______________________

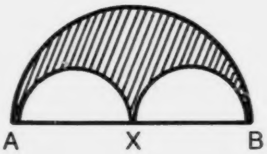
$m\,\overset{\frown}{AX}$ = ______________________

$m\,\overset{\frown}{XB}$ = ______________________

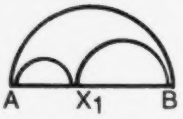
Since $R = r_1 + r_2$

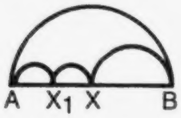
then πR = ____________________

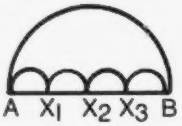
and $m\,\overset{\frown}{AB}$ = ____________________

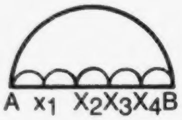


Consider points X_1, X_2, X_3, X_4, . . . to be points between A and B in the following:





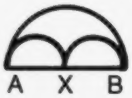




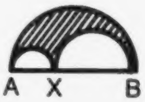
What can you predict about the relationship between these semicircular arcs? ______________________________

Find a formula to express your prediction in terms of r_1, r_2, . . .:

____



Suppose r_1 is 5 and r_2 is 8. Does $\widehat{AB}$ equal 13 π? ______________
Try your formula using $r_1 = 16$ and $r_2 = 4$.
How would you find the area of the arbelos?



One formula for finding the area of the arbelos is $\pi r_1 r_2$ when given the figure and values for r_1, r_2. Does your formula match? If not, can you see why it would? ______________________________

____

____

Use $r_1 = 5$ and $r_2 = 8$ in your formula and $\pi r_1 r_2$. Hopefully, both formulas should equal a value of 40 π square units.

Try the formula $\pi r_1 r_2$ with $r_1 = 16$ and $r_2 = 4$.

____

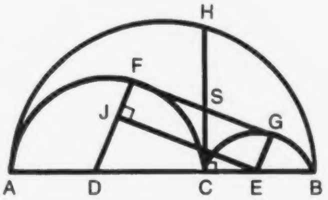
You may try the two formulas using other measures you create.

Step 2. What other properties does an arbelos have? Let's find them. In the following figure,

$\overline{HC} \perp \overline{AB}$ at C

$\overline{FG}$ is a common external tangent of the two smaller semicircles

$\overline{JE} \perp \overline{FD}$



Since $\overline{HC}$ is a line segment drawn perpendicular to the diameter $\overline{AB}$ then HC is the geometric mean between the segments of the diameter or

$$\frac{AC}{HC} = \frac{HC}{CB} \text{ or } \frac{2r_1}{HC} = \frac{HC}{2r_2}$$

or $(HC)^2$ is ____________________

Also FG = JE because ____________________

Write JD and DE in terms of r_1 and r_2 ____________________

JD = ____________________ DE = ____________________

Find $(JE)^2$ in terms of r_1 and r_2 in ΔDJE:

$(JE)^2$ ____________________

Since JE = FG

and $(JE)^2 = (FG)^2$ then $(FG)^2$ = ____________________

$\therefore (HC)^2 = (FG)^2$ and ____________________

Determine that $\overline{HC}$ and $\overline{FG}$ bisect each other at S: ____________________

Step 3. In Step 2 above, you found that $\overline{HC}$ and $\overline{FG}$ bisect each other at S. Consider the points F, H, G, C, and point S. Could these points form a circle? Why?

Determine whether the area of the circle with center at S and a diameter of HC equals the area of the arbelos or not. (Hint: From Step 1 you found the area of the arbelos to be $\pi r_1 r_2$).

Worksheet Answers and Explanations

Step 1. The arc length of a semicircle can be found by using ½ $(2\pi r)$ or πr.
$m\,\widehat{AB} = \pi R$
$m\,\widehat{AX} = \pi r_1$
$m\,\widehat{XB} = \pi r_2$
$\pi R = \pi (r_1 + r_2)$
$m\,\widehat{AB} = m\,\widehat{AX} + m\,\widehat{XB}$

The arc length of $\widehat{AB}$ equals the sum of the arc lengths of the smaller semicircles formed on $\overline{AB}$.

$m\,\widehat{AB} = \pi (r_1 + r_2 + r_3 + \ldots)$

If $r_1 = 5$, $r_2 = 8$, then $m\,\widehat{AB} = 13\pi$.

If $r_1 = 16$, $r_2 = 4$, then $m\,\widehat{AB} = 20\pi$.

The area of the arbelos will be found by subtracting the sum of the areas of the smaller semi-circles from the area of the semi-circle formed by $\overline{AB}$. Students will find this easy to understand. The formula can be derived by:

Area of the arbelos $= \text{area}\underset{A\qquad B}{\frown} - \left(\text{area}\underset{A\qquad X}{\frown} + \text{area}\underset{X\qquad B}{\frown}\right)$

$$= \tfrac{1}{2}\pi R^2 - \left(\tfrac{1}{2}\pi r_1^2 + \tfrac{1}{2}\pi r_2^2\right)$$

$$= \tfrac{1}{2}\pi R^2 - \frac{\pi r_1^2}{2} - \frac{\pi r_2^2}{2}$$

$$= \tfrac{1}{2}\pi (r_1 + r_2)^2 - \frac{\pi r_1^2}{2} - \frac{\pi r_2^2}{2}$$

$$= \tfrac{1}{2}\pi (r_1^2 + 2r_1r_2 + r_2^2) - \frac{\pi r_1^2}{2} - \frac{\pi r_2^2}{2}$$

and $= \pi r_1 r_2$

If $r_1 = 5$, $r_2 = 8$, then the area of the arbelos $= 40\pi$.

If $r_1 = 16$, $r_2 = 4$, then the area of the arbelos $= 64\pi$.

Step 2. $(HC)^2 = 4r_1r_2$

FG = JE because FGEJ is a rectangle

Since $FD = FJ + JD$
Then $JD = FD - FJ$
$= r_1 - r_2$
Since $DE = DC + CE$
Then $DE = r_1 + r_2$
In ΔDJE with right angle, $\angle DJE$
$(JE)^2 = (DE)^2 - (JD)^2$ or $4r_1r_2$
Thus, $HC = FG = 2\sqrt{r_1r_2}$
Since $\overline{SC}$ is a common internal tangent to both circles,
FS = SC and SC = SG.
Thus, FS = SG
Remember FG = FS + SG, FG = HC, and HC = HS + SC
So FS + SG = HS + SC
Hence SG = HS
Then FS = SG = HS = SC

Step 3. Since points F, H, G, C are equidistant from point S, these points form a circle with center at S.

The area of circle S $= \pi r^2$

$= \pi (SC)^2$

$= \pi (\sqrt{r_1 r_2})^2$

$= \pi r_1 r_2$

Note:

diameter

$HC = 2\sqrt{r_1 r_2}$

radius

$SC = \frac{1}{2} HC$

$= \sqrt{r_1 r_2}$

The area of the arbelos ($\pi r_1 r_2$) equals the area of circle S.

EXTENDED LEARNING:

1. Students might want to develop or create problems for each other using different values for r_1 and r_2.
2. Students might want to create problems for each other using differing values for more than two enclosed semicircles.
3. Other problems can be found in Martin Gardner's "Mathematical Games" in the *Scientific American* (January 1979, Volume 240, No. 1, pp. 18–28).
4. An extension can be found in Alfred Posamentier's *Teaching Secondary School Mathematics* (pp. 292–293).

REFERENCES:

Posamentier, Alfred S. and Sheridan, Gordon. *Math Motivators: Investigations in Geometry, Algebra, Pre-Algebra.* Menlo Park, CA: Addison-Wesley, 1982, 1983.

Posamentier, Alfred S. and Stepelma, Jay. *Teaching Secondary School Mathematics.* Columbus, Ohio: Merrill Publications, 1981.

Eves, Howard. *A Survey of Geometry.* Vol. 1. Boston: Allyn and Bacon, Inc., 1972.

AUSTRIAN SUBTRACTION

GRADE LEVEL/ COURSE: 4–5

INTRODUCTION/ OBJECTIVE: The student will use Austrian Subtraction to solve problems of three or more digits with regrouping.

PROCEDURE: This large group activity provides an alternate method of subtraction for students who have mastered basic subtraction skills.

ACTIVITY:

Step 1. Explain to the class that this method of subtraction involves finding the number which needs to be added to the subtrahend to get the top number. For example, what needs to be added to 185,629 to get 1,349,286:

$$\begin{array}{r} 1{,}349{,}286 \\ -\ \ 185{,}629 \\ \hline \end{array}$$

Because we cannot add a positive integer to 9 to get 6, we put in a 1 to get 16. We "balance" this by also adding a 1 to the integer to the left of the 9. Thus, the 2 becomes a 3:

$$\begin{array}{ccccccc} 1 & 3 & 4 & 9 & 2 & 8 & {}^{1}6 \\ - & 1 & 8 & 5 & 6 & \text{``3''} & 9 \\ \hline & & & & & 5 & 7 \end{array}$$

→ 7 + 9 = 16

→ 5 + (2 + 1) = 5 + 3 = 8

Step 2. Continue with this procedure demonstrating the complete solution:

Original problem

$$\begin{array}{r} 1{,}349{,}286 \\ -\ \ 185{,}629 \\ \hline \end{array}$$

$$\begin{array}{ccccccc} 1 & 3 & {}^{1}4 & 9 & {}^{1}2 & 8 & {}^{1}6 \\ & 1^{+1} & 8 & 5^{+1} & 6 & 2^{+1} & 9 \\ \hline 1 & 1 & 6 & 3 & 6 & 5 & 7 \end{array}$$

→ 7 + 9 = 16

→ 5 + (2 + 1) = 5 + 3 = 8

→ 6 + 6 = 12

→ 3 + (5 + 1) = 3 + 6 = 9

→ 6 + 8 = 14

→ 1 + (1 + 1) = 1 + 2 = 3

→ 1 + 0 = 1

Step 3. Include several other examples in a chalkboard demonstration.

Step 4. Have the students work the following problems using the Austrian method:

$$\begin{array}{r} 765342 \\ -549239 \\ \hline \end{array} \qquad \begin{array}{r} 6344273 \\ -1496259 \\ \hline \end{array} \qquad \begin{array}{r} 403001 \\ -\ \ 59353 \\ \hline \end{array}$$

$$\begin{array}{r} 23105 \\ -10706 \\ \hline \end{array} \qquad \begin{array}{r} 200496 \\ -128747 \\ \hline \end{array}$$

*The teacher should make the point that Austrian subtraction works because the same amount is added both to the minuend and subtrahend of the problem.

THE BASEBALL PROBABILITY DIAMOND

GRADE LEVEL/ COURSE: Advanced Mathematics

INTRODUCTION/ OBJECTIVE:

1. Students will use Pascal's triangle to determine the odds that a team will win the World Series.
2. Students will be able to demonstrate the changes that occur in the odds as each game is completed.
3. Students will be able to distinguish between the probabilities of a team winning when both teams are evenly matched and when one team has a decided advantage.

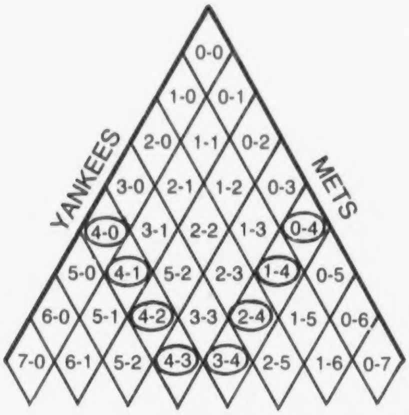
This topic is also suitable for students in the 7th through 9th grades who have had some exposure to probability. In addition to expanding their understanding of probability, it provides valuable practice in working with fractions.

Have students consider the following:

> If the often talked about "Subway" World Series between the Mets and the Yankees were to take place, what are the possible final "won-lost" outcomes for each team?

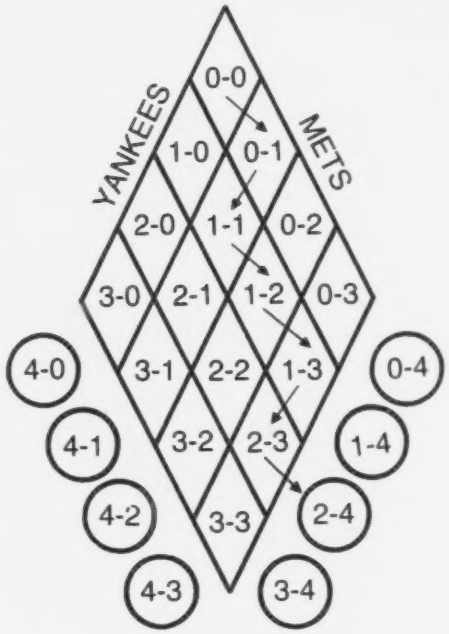
PROCEDURE: After reviewing the outcomes listed by the students, summarize as follows —

Since as many as 7 games can be played, the following diagram illustrates each of the possible outcomes after every day's play is completed. The triangle that is formed is an example of Pascal's triangle, named for the 17th century French mathematician, Blaise Pascal.



Elicit that since a team needs to win 4 games in order to win the series, the lower left and right corners of the triangle do not apply since these spaces represent games that would not be played. The won-lost situation inside the circles represent possible final outcomes. Therefore, as shown in the following diagram, Pascal's Triangle becomes —

PASCAL'S BASEBALL DIAMOND



As we will see later, there are 70 possible ways to play the series. Have students follow several of the 70 possible routes through the diamond. (one possible route is demonstrated by the arrows in the above diagram).

ACTIVITY 1: Discuss the factors which would contribute to a team's winning or losing.

Elicit that a wide range of factors are involved — from the talents of the players to the weather conditions, whether or not the team is playing at home, the conditions of the pitcher's elbow, or if the team's best hitter had a fight with his wife before the game.

In our analysis of the series, we will simplify the situation by assuming that the Mets and the Yankees are evenly matched — both in talent and conditions — making the winner of each game a "toss-up."

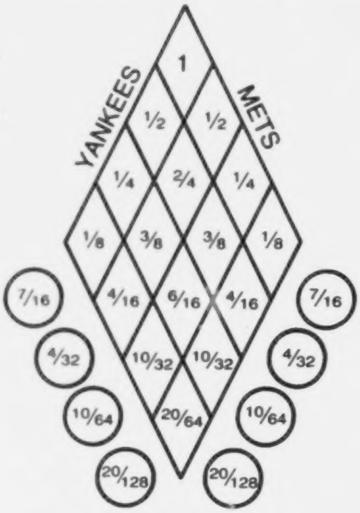
Working under this assumption, ask the students:

Before the first game begins, what is the probability the Mets will win
- (i) the first game?
- (ii) the first 2 games?
- (iii) the first 3 games?
- (iv) the first 4 games?

The Pascal's Probability Diamond at the right shows the probability that corresponds to the won-lost space in the diamond shown on the previous page.

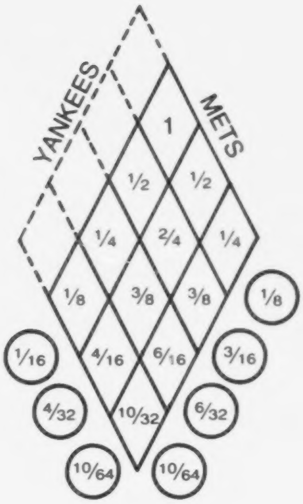
Have students note that these probabilities apply only before the first game is played. Students should also be aware that the rule for obtaining the probabilities that are inside the circles is changed because the Series ends when one team has won 4 games (the probability in a circle does not contribute to the probability shown in the circle before it). For this reason most of the circled probabilities are not the same as those shown in the corresponding diamond-shaped spaces of a full Pascal's triangle.

BEFORE FIRST GAME

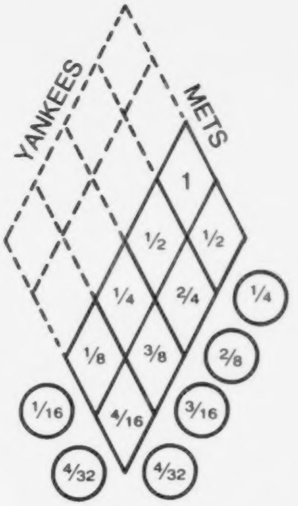


Now suppose that the Mets have won the first game — the probabilities for all the remaining possibilities must be recalculated starting with a 1 in the Yankees 0–Mets 1 space. If the Mets were to win the first 2 games, a 1 would be placed in the 0-2 space. The calculations for other possibilities would be made in a similar fashion.

YANKEES BEHIND 0-1



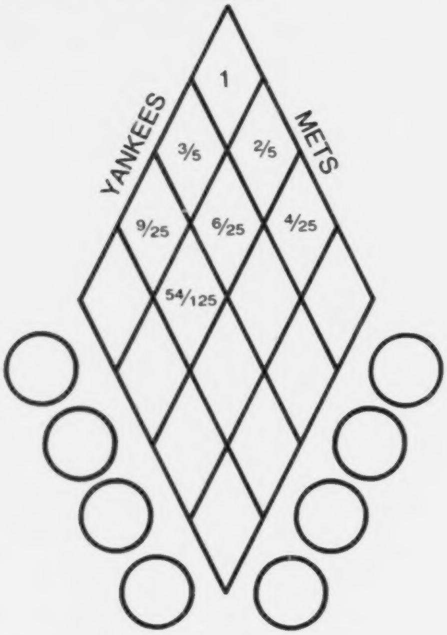
YANKEES BEHIND 0-2



It should be noted that in each of the 3 situations above the sum of all the probabilities in the circles is equal to 1. This indicates that one way or another the Series is certain to be completed. The sum of all circled probabilities on each side of the diamond is that team's probability for winning. The number of different ways the remaining games can be won and lost is found by finding the sum of the numerators of the circled fractions. Therefore there are 70 ways to play the Series starting before the game, 35 ways to play the Series after one game has been played, etc.

ACTIVITY 2: Now, examine the case where the Mets and the Yankees are not evely matched. I suppose the oddsmakers have determined that the probability that the Yankees will win the Series is 3/5 and the probability that the Mets will win the Series is 2/5. In filling in Pascal's Diamond, instead of taking 1/2 of each of the probabilities in the two spaces above, you would take 3/5 of the number above and to the right of the space and 2/5 of the number above and to the left of the space.

Have students complete the diamond —



EXTENDED LEARNING: Using Pascal's Probability Diamond, find the probabilities of each team winning if the oddsmakers say the probability that the Mets will win is 2/3 and the probability that the Yankees will win is 1/3.

A. before the first game has been played

B. if the Mets have won the first and the Yankees won the second

C. if the Mets have won the first two and the Yankees have won the third.
(one Diamond to be used for each of the 3 situations)

REFERENCES:

Kadesch, Robert R. *Math Menagerie.* New York: Harper and Row, 1970.

Dantzig, Tolia. *Number, the Language of Science.* New York: MacMillan Company, 1930.

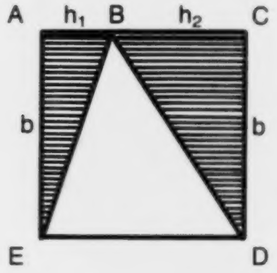
BELIEVING THAT IT CAN BE SOLVED

GRADE LEVEL/ COURSE: 7–8

INTRODUCTION/ OBJECTIVE: The student will be able to attack a problem by assuming that it is solvable and then drawing conclusions from that assumption.

PROCEDURE: The teacher will present the following activities in class.

ACTIVITY 1: Post a large copy of the diagram below and discuss the problem. Allow students adequate time to attempt to solve the problem, suggesting that they keep the answer secret, revealing it only to you.

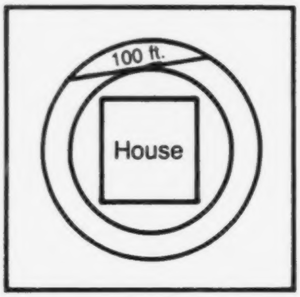


The sides of square ACDE
measure 12 cm. What is the
area of the shaded region?

Some students may protest that not enough information is given, because the heights of the shaded triangles are not given. Encourage students to believe that the problem is solvable and to continue their attempts. If necessary, guide students in self-questioning techniques. What if it *is* solvable? *Then* what? What must the answer depend on? What are we looking for? (Area, and the area of the triangles depends on two linear measures, b and h. We have $h_1 + h_2 = s$, so the lengths of the heights must depend on the number 12.) If necessary, suggest that students seek an indirect solution. (Find the area of the unshaded portion and subtract it from the area of the rectangle.) Discuss the effect of assuming a problem is unsolvable or solvable in life in general. Compare and contrast such assumptions in mathematics with those in life in general. Answer: The area of the shaded region is 72 square cm, one half the area of the square.

ACTIVITY 2: Present the diagram below and discuss the problem.

A young girl once aspired to become a world class distance swimmer. She wished for a pool of her own in which to build her endurance, but her family lived in a house on a small lot only a little larger than their house. The only way she could swim long, uninterrupted distances would be to build a ring-shaped pool encircling the house. She drew the plan below and asked her parents if she could have such a pool with a depth of six feet.



They told her she could have her pool if she could figure how much water it would hold. Can you find the volume of the pool? (Note: The chord is tangent to the inner circle.)

This problem may take a great deal of time, even weeks or months if no guidance is given. Students may insist that not enough information is given. Remind them of the problem in Activity 1 and the problem-attack strategies discussed. Later (maybe days later) give more specific hints such as the following:

1. Assume that the problem is solvable.
2. Assume that we have enough information.
3. Then the answer must *not* depend on ____________.
4. How can we eliminate it or make it zero? What effect will that have?
5. The answer depends only on ___?___ and ___?___.

Answers:

3. The diameter of the inner circle. (Note that it is the length of the chord that must remain constant, therefore the diameter of the outer circle must be $\geq$ 100).
4. Shrink the center circle to a point (a circle with diameter zero). and shrink the outer circle so that the chord remains 100 feet. This causes the outer circle to have a diameter equal to the given chord.
5. The *length* of the *chord* and the *depth* of the pool.

The solution in cubic feet is

$$\begin{aligned} V &= Bh \\ &= (\pi r^2)h \\ &= (\pi 50^2)(6) \\ &= 15{,}000\,\pi \\ &= 47123.85 \text{ cu. ft. of water} \end{aligned}$$

,where B = the area of the ring
and h = the depth of the pool

REFERENCES:

Gardner, Martin. *Aha! Insight!*. New York: W. H. Freeman and Company, 1978.

BISECTING A LINE SEGMENT

GRADE LEVEL/ COURSE: Geometry

MATERIALS: Compass and straightedge.

INTRODUCTION/ OBJECTIVE: Students will bisect a line segment in two or more ways.

Most geometry students only know one way to bisect a line segmet. Most of them should be interested in the variety of ways this construction can be done. Beginning students can just be taught the constructions. More advanced students can prove that the constructions do bisect a line segment.

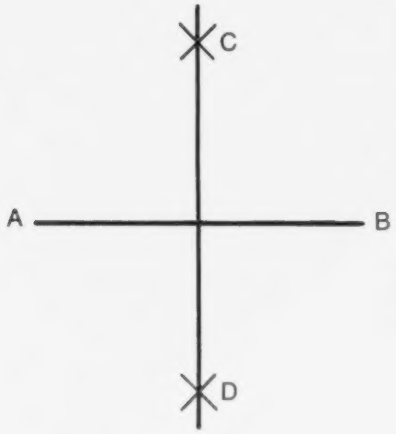
Students should know the basic geometric constructions usually taught in high school. However, the constructions below should prove interesting to any student who has learned to bisect a line segment.

PROCEDURE: Greek geometry uses only the compass and an unmarked straightedge in construction. Although the constructions usually taught in high school use a modern compass and straightedge, they can be done with the Greek compass, though not always as easily. The constructions that follow use a variety of tools: compass only, ruler with parallel edges, ruler with fixed angle, straightedge, and straightedge with two marks.

ACTIVITY 1: The first diagram shows the usual method for bisecting a line segment using a straightedge and compass. Use the following steps in the construction.

Step 1. Given $\overline{AB}$, use A as a center and draw arcs above and below line segment $\overline{AB}$. These arcs should have a radius greater than ½ the length of $\overline{AB}$.

Step 2. Use B as a center and the same radius, and draw arcs that intersect those arcs already drawn. Label the intersecting points C and D.



Step 3. Using the straight edge, draw line segments CD. CD bisects AB.

ACTIVITY 2: In 1797, Mascheroni, an Italian mathematician, demonstrated that all the Greek geometric constructions could be done with compasses alone. To effect this construction, follow these steps:

Step 1. Given a segment AB, construct a circle whose center is A and whose radius is AB.

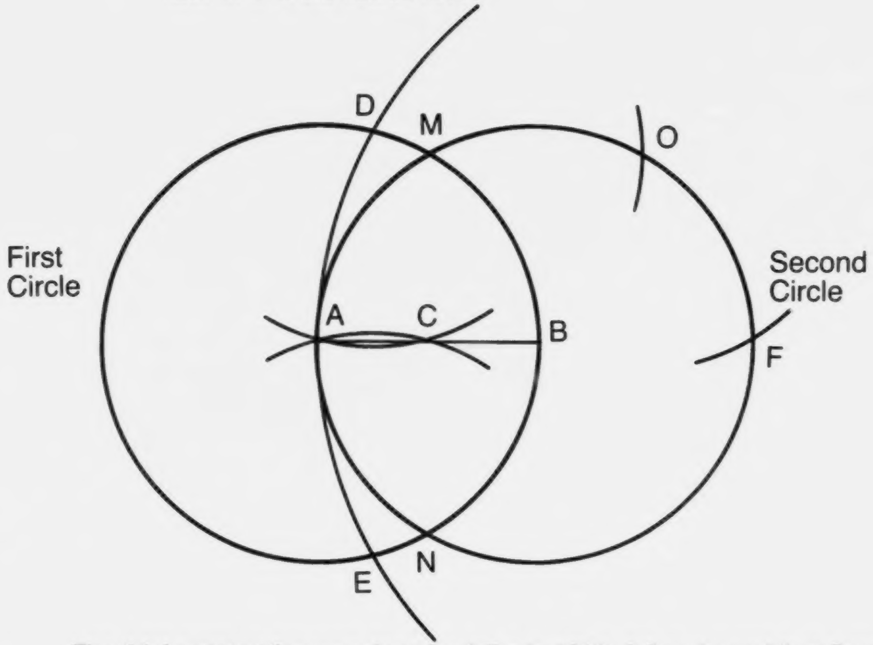
Step 2. Now, construct a circle whose center is B and whose radius is AB. Let the two points of intersection be called M and N.

Step 3. Draw an arc with a center at M and a radius of AB to intersect the second circle at O.

Step 4. Use the same radius and a center O to draw an arc that intersects the second circle at F.

Step 5. Using F as the center and a radius of FA draw an arc which intersects the first circle at D and E.

Step 6. Using centers D and E and a radius of AB, draw arcs that intersect at point A and C. C is the middle of $\overline{AB}$.



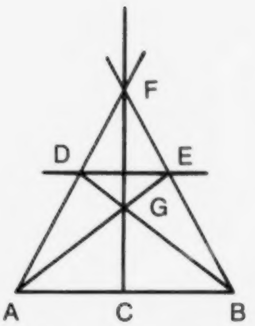
ACTIVITY 3: The third construction was done by Adler in 1890. Adler showed that Greek constructions could be drawn using only a ruler with two parallel edges. To complete this construction, proceed as follows:

Step 1. Given $\overline{AB}$, place the ruler so that one edge is on $\overline{AB}$ and draw a line parallel to $\overline{AB}$.

Step 2. Draw two lines, one through A and one through B, that intersect the other parallel line at D and E and meet at point F.

Step 3. Next draw line segments $\overline{DB}$ and $\overline{AE}$. Label their intersection G.

Step 4. Draw $\overleftrightarrow{FG}$ to intersect $\overline{AB}$ at C. C is the midpoint of $\overline{AB}$.



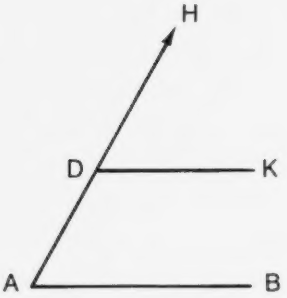
ACTIVITY 4: Adler also showed that all Greek constructions could be done with a ruler having a fixed acute or right angle. To construct a line bisector using this method, proceed as follows:

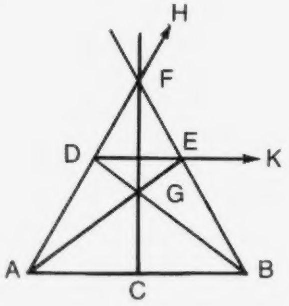
Step 1: Given AB, place the ruler with the corner having the fixed angle on A and the edge of the ruler along AB. Draw a line AF to form an angle with line AB.

Step 2. Pick any point D on the line AH and line up the ruler so that the edge having the fixed angle is on AH. Draw a line through D to form line $\overline{DK}$. $\overline{DK}$ is parallel to AB.

Step 3. Draw a segment from point B to some point F on line AH such that D is between A and F. Label the intersection of $\overline{BF}$ and $\overline{DK}$ point E.

Step 4. Use steps 3 and 4 from Activity 3 above.





ACTIVITY 5: Steiner in 1833 showed that all the Greek constructions could be done with a straightedge if given a circle and its center. The steps of construction follow:

Step 1. Given $\overline{AB}$ and a circle whose center (O) is not on $\overline{AB}$, draw segments from B through O to intersect the circle at P and Q, and from A through O to intersect the circle at R and S.

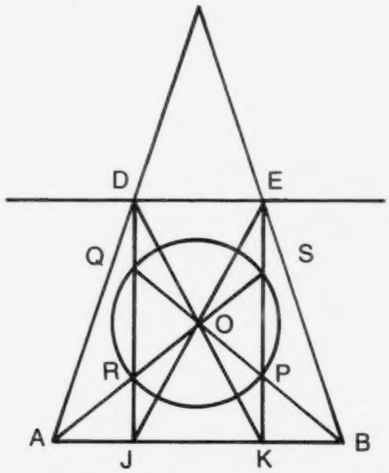
Step 2. Draw $\overleftrightarrow{QR}$ and $\overleftrightarrow{SP}$ to intersect $\overline{AB}$ at J and K respectively.

Step 3. Draw $\overleftrightarrow{KO}$ to intersect $\overleftrightarrow{QR}$ at D, and $\overleftrightarrow{JO}$ to intersect $\overleftrightarrow{SP}$ at E.

Step 4. Draw $\overleftrightarrow{DE}$. $\overleftrightarrow{DE}$ is parallel to $\overline{AB}$.

Step 5. Extend $\overline{AD}$ and $\overline{BE}$ to intersect at F.

Step 6. Follow steps 3 and 4 from Activity 3 above.



ACTIVITY 6: This last construction uses a straightedge marked in two places. Markings were not permitted in the classical Greek constructions, but the Greeks did study constructions made with marked straightedges. Following are the steps for constructing a line bisector using a straightedge having two marks on it:

Step 1. Given $\overline{AB}$, draw a line through A at an angle to $\overline{AB}$.

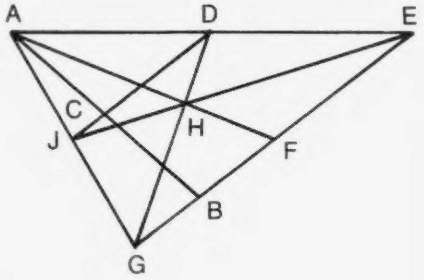
Step 2. Use the marked straightedge to locate D and E on the line so that AD = DE. (AD is the distance between the two marks on the straightedge.)

Step 3. Draw EB and locate F and G on it, so that EF = FG. (Also the distance between the marks.)

Step 4. Draw $\overline{GD}$ and $\overline{AF}$ to intersect at H.

Step 5. Draw $\overline{EH}$ and $\overline{AG}$ to intersect at J.

Step 6. Draw $\overline{DJ}$ to intersect $\overline{AB}$ at C. C is the midpoint of $\overline{AB}$.



EXTENDED LEARNING: Students will bisect a given line in two different ways.

REFERENCE: Gibb, Allan A. "There are More Ways Than One to Bisect a Line Segment." *Mathematics Teacher.* May 1977.

CASTING OUT NINES

GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVES: The student will learn an alternative method for checking long addition problems by casting out nines.

PROCEDURE: This is a concept which may be taught to small groups or the entire class when the students have mastered the grade level objectives in addition.

ACTIVITY: The teacher should proceed in the following manner to demonstrate casting out nines:

Step 1. Suppose you want to check the following problem:

$$\begin{array}{r} 534 \\ 721 \\ 981 \\ 787 \\ +\ 430 \\ \hline 3453 \end{array}$$

First add the digits in each horizontal row dropping out the 9s. In the first row, 534, you can see that dropping the 5 and 4 that make nine leaves only 3. So in a separate column, write 3:

534 3

The first three rows will look like this:

534	3	
721	1	(because 7 + 2 = 9)
981	0	(drop the 9, and 8 + 1 = 9)

Step 2. What about the next line, 787?

7 + 8 + 7 = 22; 2 + 2 = 4.

$$\begin{array}{r} 534 \\ 721 \\ 981 \\ 787 \\ +\ 430 \\ \hline 3453 \end{array} \qquad \begin{array}{r} 3 \\ 1 \\ 0 \\ 4 \\ +\ 7 \\ \hline 15 \end{array}$$

When you have completed this process for all addends, add the numbers in the new column. Then add the digits of both sums, again casting out nines.

Step 3. Compare the results.

6 = 6
Right? Right!

Step 4. Let's try another one

$$\begin{array}{r@{\qquad\qquad}r} 630 & 0 \\ 754 & 7 \\ 326 & 2 \\ 985 & 4 \\ +\ \ 273 & +\ \ 3 \\ \hline 2968 & 16 \end{array}$$

$$7 = 7$$

Step 5. Give the students additional problems to work. Have them check by casting out nines.

CHALLENGING COUNTING PROBLEMS

GRADE LEVEL/ COURSE: Advanced Mathematics

INTRODUCTION/ OBJECTIVE: Students will be able to solve challenging counting problems which have non-standard solutions, including ring permutations, circular permutations, and others. Students should be able to find similarities in different-looking problems. (In many cases counting problems cannot be solved by using the basic formulas for n!, nPr, nCr, $\frac{n}{n_1!\,n_2!\,\ldots\,n_k!}$ etc).

Students should be familiar with, and be able to solve, standard problems involving:

1. permutations
2. combinations
3. arrangements of objects, some of which are identical
4. the number of subsets of an n-element set.

PROCEDURE: Work with the entire class, directing class discussion through questioning.

ACTIVITY 1:

Step 1. Present a problem which is similar to, but more involved than problems which have previously been discussed. For example: "In how many ways may 4 horses be arranged in a merry-go-round?" The students' response will be probably be 4!, which would be incorrect. To demonstrate, 4 student volunteers should come to the front of the room and arrange themselves in a circle. An arrangement might be:

Lenny
Benny Penny.
Denny

The class can state the arrangement: "Lenny, Penny, Denny, Benny".

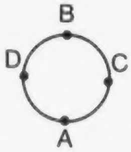
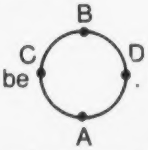
Step 2. Now each student should move one place, say clockwise, giving

Benny
Denny Lenny.
Penny

Again the class can state the arrangement: "Lenny, Penny, Denny, Benny". The students should quickly realize that this arrangement is no different than the first.

Step 3. Then, the 4 students should move, once again, one position clockwise. Again, the class should realize that this is not a new arrangement. Hopefully by now the class will see that there are 4 such identical arrangements, and that the answer to the original question must be $\frac{4!}{4} = 3!$, or 6.

Step 4. Then, the same question can be asked for 5 horses, giving the result 4! or 24. This leads to the generalization that the number of ways n objects can be arranged in a circle is $(n-1)!$. That is, each circular permutation produces n linear permutations (or $nCPn = \frac{n!}{n} = (n-1)!$, where CP represents "circular permutations").

ACTIVITY 2: An important related problem is "In how many ways may 4 beads be strung on a ring?". Some students should quickly answer (4 − 1)! or 3! or 6. They will probably be surprised to learn that their answer is not correct. This can be shown by a simple diagram. One possible arrangement of the 4 beads is:  "Another" arrangement might be . However, these two arrangements are, in fact, the same; since the ring lies in space, it can be "flipped over". So, we can obtain the "second" arrangement from the first merely by flipping over the ring, without rearranging the beads. Therefore, a clockwise orientation is the same as a counter-clockwise orientation. Thus, each "ring permutation" produced 2n linear permutations, and the correct solution is $\frac{(4-1)!}{2}$ or $\frac{3!}{2}$ or 3. In general: $nRPn = \frac{n!}{2n} = \frac{(n-1)!}{2}$ where RP represents "ring permutations".

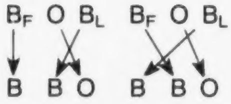
ACTIVITY 3: Several more problems should be done, to be sure that students can differentiate between circular and ring permutations. For example:

1. In how many ways may 7 people be seated around a table?
   (Circular permutation: (7 − 1)! or 6! or 720)
2. In how many ways may 6 charms be arranged on a bracelet?
   (Ring permutation: $\frac{(6-1)!}{2}$ or $\frac{5!}{2}$ or 60)

ACTIVITY 4:

Step 1. Another interesting type of problem involves arrangements of objects, some of which are identical. For example, a typical problem would be: "In how many ways may the letters of STATISTICS be arranged?". Based on their previous work, the students should be able to come up with $\frac{10!}{3!\,3!\,2!}$ the correct answer.

Step 2. Once that is agreed upon, the students can go to: "In how many ways may the letters STATISTICS be arranged, if all the s's must occur together?". To analyze this, it may be necessary to consider a simpler word: BOB (for the same question, referring to both B's). The first B might be called B_F and the second B_L.

A diagram might help:  Showing the students that putting the first B first and the second B second or the first B second and the second B first gives the same "word", i.e., arrangement, hopefully, will lead to at least one or two students seeing that the two B's may be considered as one letter ("BB").

Step 3. It may be necessary to propose another problem here: suppose, in the Martian alphabet, two of the letters are BB and O, and suppose that BBO is a two-letter Martian word. In how many ways may the letters of BBO be arranged? By showing that, 1) there are only 2 such permutations and b) there are only 2 permutations of BOB with the B's occurring together. The students should then see that a) these problems are identical and b) therefore, the way to solve the original problem is to consider all 3 S's as one letter.

Thus, the solution to the original problem is $\frac{8!}{3!\,2!}$, since STATISTICS now is considered to have 8 letters (the SSS and 7 others, with 3 T's and 2 I's). After solving several problems (using words such as TENNESSEE, CALCULUS, etc.), a nice, very challenging problem would be: "In how many ways may the letters of STATISTICS be arranged, with exactly 2 of the 3 S's together?". This is a very difficult problem, which may be better suited to be an extra-credit, or optional, homework problem. Some students may correctly reason that the way to solve this problem is to take the number of ways STATISTICS may be arranged with 2 S's together (remember, this includes the cases where the third S is adjacent to the other 2, giving all 3 S's together) and subtract the number of arrangements with all 3 S's together. Thus, their answer would be:

$\frac{9!}{3!\,2!} - \frac{8!}{3!\,2!}$. Unfortunately, this is not quite correct. You, must take into account the 2 ways that the 3 S's may occur together: (SS)S and S(SS). Therefore, the correct solution is: $\frac{9!}{3!\,2!} - 2\left(\frac{8!}{3!\,2!}\right) = \frac{9 \cdot 8! - 2 \cdot 8!}{3!\,2!} = \frac{7 \cdot 8!}{3!\,2!} = 23{,}520$.

Step 6. If some students need convincing that the additional factor 2 is necessary, the same question may be answered for the word SASS. In this case there are few enough arrangements that they may conveniently be listed, and shown to fit the above "formula".

For SASS, there are only 2 arrangements with exactly 2 of the 3 S's together: SASS and SSAS. The "formula" holds since:

3! (number of arrangements with 2 S's together; i.e., same as 3 letters)	−	2! (number of arrangements with all 3 S's together; i.e., same as 2 letters, multiplied by 2)	= 2

ACTIVITY 5: There are also interesting problems which involve combinations. A standard problem is: "Cal Q. Lus has 10 friends. In how many ways may he invite 6 of them to a party?" The simple solution is ${}_{10}C_6$ or $\frac{10!}{6!\,4!}$ or 210, since the order in which he invites them is irrelevant. There are many challenging variations of this problem. Such as the following examples:

1. In how many ways may Cal invite some, or all, of his friends? Many students should be able to answer by the "peasant's method": , ${}_{10}C_1 + {}_{10}C_2 + {}_{10}C_3 + \ldots + {}_{10}C_{10}$, since Cal may invite 1 of the 10, or 2 of the 10, or 3 of the 10, and so on, up to 10 of the 10. However, based on previously discussed material, some students should also come up with the "poet's method": $2^{10} - 1$, since this is equal to the number of non-empty subsets of a 10-element set (which is really what the question asks).

2. In how many ways may Cal invite 6 of his friends, one on each of 3 successive Mondays? In this case, the solution is a permutation, since the order, i.e. arrangement, that Cal chooses makes a difference. So the solution would be ${}_{10}P_6$ or $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ or 151,200.

3. In how many ways may Cal invite some, or all, of his friends, if 3 are inseparable?

   With a little thought, it should be obvious that since these 3 friends are inseparable, Cal must either invite all 3 of them, or none of them. With a little more thought, it should be equally obvious that this problem is analagous to arranging the letters of STATISTICS with all 3 S's together. Therefore, to arrive at the solution one must consider the 3 inseparable friends as one "friend", and the problem then becomes:

"Cal Q. Lus has 8 friends (the "triple-header" and the remaining 7). In how many ways may he invite some, or all, of them?" The solution is now no longer difficult: $2^8 - 1$ or 255.

4. In how many ways may Cal invite some, or all, of his friends, if 3 hate each other?

   Assuming Cal wishes to avoid a confrontation, he has 2 choices—he may invite none of these 3, or at most one of these 3. If he invites none of these 3, he must invite at least one of the remaining 7 friends, which can be done in $2^7 - 1$ ways. If he invites one of these 3 (which, of course, can be done in 3 ways), he may then invite none, some, or all of the others (which can be done in 2^7 ways). Therefore, the solution must be: $2^7 - 1 + 3(2^7)$.

EXTENDED LEARNING:

1. Explain the difference between the following types of permutations (giving an example of each): linear, circular, ring.
2. In how many ways may the letters of BOOBOO be arranged:
   a) with no restrictions?
   b) with both B's together?
   c) with all 4 O's together?
   d) with both B's and all 4 O's together?
   e) with exactly 3 of the 4 O's together?
3. Dooby Gud has 6 friends: 4 Capitalists and 2 Communists. In how many ways may he invite a guest(s) to dinner if he wants:
   a) at least one guest?
   b) exactly 3 guests?
   c) at least 3 guests?
   d) at least one member of each party?
   e) an equal number of each party?
   f) all or none of each party?
   g) only guests of one party?

REFERENCES:

Peters and Schaaf. *Algebra and Trigonometry—A Modern Approach.* Princeton, N.J.: Van Nostrand, 1965.

Edgerton and Carpenter. *Advanced Algebra.* Boston: Allyn and Bacon, 1961.

Hall and Knight. *Higher Algebra.* London: MacMillan, 1960.

CHANCES OF WINNING A LOTTERY WITH A BUCK

GRADE LEVEL/ COURSE: Advanced Mathematics or at the end of a probability unit because it ties together most of the concepts taught in this section.

INTRODUCTION/ OBJECTIVE:

1. Students will calculate the number of combinations of n elements taken r at a time.
2. Students will calculate the number of combinations formed from more than one set.
3. Students will calculate the probability of independent events occurring.
4. Students will calculate the probability that a specific outcome occurs at least once.

PROCEDURE:

1. Students should be familiar with the definitions of sample space, permutation, complement, compound events, P(A and B), P(A or B), and combination.
2. The following problem would be distributed to the students at the beginning of the class:

ACTIVITY:

Step 1. (a) A lottery card can be purchased for $1.00 and it has two separate games. There is a set of numbers from 1 to 10 on each game and six numbers must be selected from each set.

YOU WIN THE:	IF:
GRAND PRIZE	six selected numbers are the same as those drawn
SECOND PRIZE	five of the selected numbers are the same as those drawn
THIRD PRIZE	four of the selected numbers are the same as those drawn

What are the probabilities of winning a prize of some type with one card?

(b) Students should be able to translate the question to read: find the union of the probabilities of winning each prize (since no two events can occur simultaneously, P(A or B or C) = P(A) + P(B) + P(C)).

(c) In order to find the probabilities of winning each prize, the sample space must be determined. Make sure that students understand the difference between combination and permutation. A permutation is any arrangement of the elements of a set in a definite order. A combination is a subset of r elements of a set with n elements (order is unimportant). This would mean the sample space is equal to ${}_{10}C_6$ (${}_{10}C_6 = 10!/6!4! = 210$).

(d) Recall that two games are played with one card. We are interested in the probability of winning at least one of these games which is 1 − P(losing both games).

Step 2: (a) **Grand Prize**

Since there is only one six-number combination that would win this prize, P(winning the grand prize in one game) = 1/210. Therefore, P(not winning the grand prize in one game) = 1 − 1/210 = 209/210. P(losing both games) = $(209/210)^2$ = 43681/44100. The loss of each game is an event which is independent. Hence, the probability of winning the grand prize i.e. of not losing both games is 1-43681/44100 = 419/44100.

(b) **Second Prize**

Five of the six numbers are the same, i.e. ${}_6C_5 = 6$.

One number of the remaining 4, i.e. ${}_4C_1 = 4$.
${}_6C_5 \cdot {}_4C_1 = 6 \cdot 4 = 24$.

P(winning 2nd prize in one game) = 24/210.

P(winning at least one of two games) = $1 - (1 - 24/210)^2 =$ 9504/44100.

(c) **Third Prize**

Four of the six numbers are the same, i.e. ${}_6C_4 = 15$.

Two of the remaining 4, i.e. ${}_4C_2 = 6$.
${}_6C_4 \cdot {}_4C_1 = 15 \cdot 6 = 90$.

P(winning 3rd prize in one game) = 90/210 = 3/7.

P(winning at least one of two games) = $1 - (1 - 3/7)^2 = 33/49$.

P(winning one of these three prizes) = 419/44100 + 9504/44100 + 33/49 = 39623/44100.

EXTENDED LEARNING:

1. Suppose that there is no grand prize winner and a seventh "spare" number is drawn. If five of the six selected numbers are the same as those drawn and the sixth selected number is the spare number, then an alternate grand prize is awarded. What is the probability of winning the alternate grand prize in at least one of the two games?
2. How would the probabilities change in a similar lottery in which the original set of 30 numbers is used?

REFERENCE: Litwiller, B. H., and Duncan, D. R. "Computing Lotto Odds". *School Science and Mathematics,* Volume 85.

CHANGING REPEATING DECIMALS TO FRACTIONS

GRADE LEVEL: 7–8

INTRODUCTION/ OBJECTIVE: The students will investigate repeating decimals to show that they are rational numbers and to explain the process of changing repeating decimals to fractions.

PROCEDURE: Begin this activity in the classroom. Assign additional exercises and extensions to be completed at home.

ACTIVITY:

		Example:
Step 1.	Let x = number of digits that repeat	1. $x = 1$
Step 2.	Let n = the repeating decimal	2. $n = .\overline{6}$
Step 3.	Multiply both sides of the second equation by 10^x ($10^1 = 10$)	3. $10n = 6.\overline{6}$
Step 4.	Subtract the second equation from the third equation	4. $10n = 6.\overline{6}$

$$\begin{array}{r} 10n = 6.\overline{6} \\ -\ \ n = .6 \\ \hline 9n = 6 \\ n = 6/9 \\ n = 2/3 \end{array}$$

Example # 1

$11/12 = .916$

1. $x = 1$
2. $n = .91\overline{6}$
3. $10n = 9.1\overline{6}$
4. $10n = 9.1\overline{6}$

$$\begin{array}{r} 10n = 9.1\overline{6} \\ -\ \ n = .91\overline{6} \\ \hline 9n = 8.25 \\ n = 8.25/9 \\ n = 825/900 \\ n = 11/12 \end{array}$$

Example # 2

$5/11 = 0.\overline{45}$

1. $x = n$
2. $n = .\overline{45}$
3. $100n = 45.\overline{45}$
4. $100n = 45.\overline{45}$
5. $-\ n = .\overline{45}$

$$\begin{array}{r} 99n = 45 \\ n = 45/99 \\ n = 5/11 \end{array}$$

Example #3

$1/7 = .\overline{142857}$

1. $x = 6$
2. $n = .\overline{142857}$
3. $1{,}000{,}000n = 142857.\overline{142857}$
4. $1{,}000{,}000n = 142857.\overline{142857}$

$$\begin{array}{rcl} 1{,}000{,}000n & = & 142857.\overline{142857} \\ - \qquad n & = & .\overline{142857} \\ \hline 999{,}999n & = & 142857 \\ n & = & 142{,}857/999{,}999 \\ n & = & 1/7 \end{array}$$

ADDITIONAL EXERCISES:

1. $n = .\overline{5}$
2. $n = .\overline{7}$
3. $n = .\overline{63}$
4. $n = .\overline{83}$
5. $n = .\overline{076923}$

EXTENDED LEARNING: Have students investigate the types of denominators that generate repeating decimals with periods of 1, 2, 3, 4, 5, and 6.

CONIC ART

GRADE LEVEL/ COURSE: Algebra II

MATERIALS: graph paper, pencils and/or colored pencils, calculator

INTRODUCTION/ OBJECTIVE: This activity is designed to promote interest in conic sections and their equations by having students produce pictures using at least four different conic sections.

PROCEDURE:

1. The student may work individually, with a partner, or in a group to produce a preliminary picture during an average class period.
2. The students will finish the project at home.
3. Three examples of the process to be followed are included.

ACTIVITY:

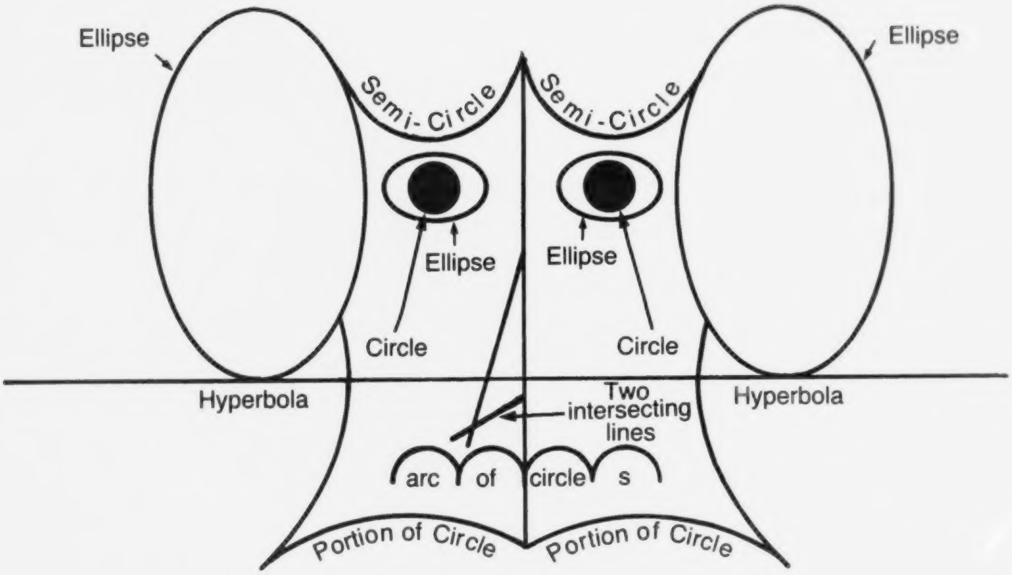
Step 1. Review the following examples of conic art on the next few pages.

Step 2. Create your own sketch using the following guidelines:

- a. Use at least four conic sections to make a rough sketch. Be sure that every part of your sketch is a conic section or a portion of a conic section.
- b. Label each conic section shown in your sketch.
- c. Produce a finished drawing on graph paper.
- d. On a separate sheet of paper, define each conic section used in your drawing in terms of X and Y.
- e. Name your creation.

Solution: **Example 1**

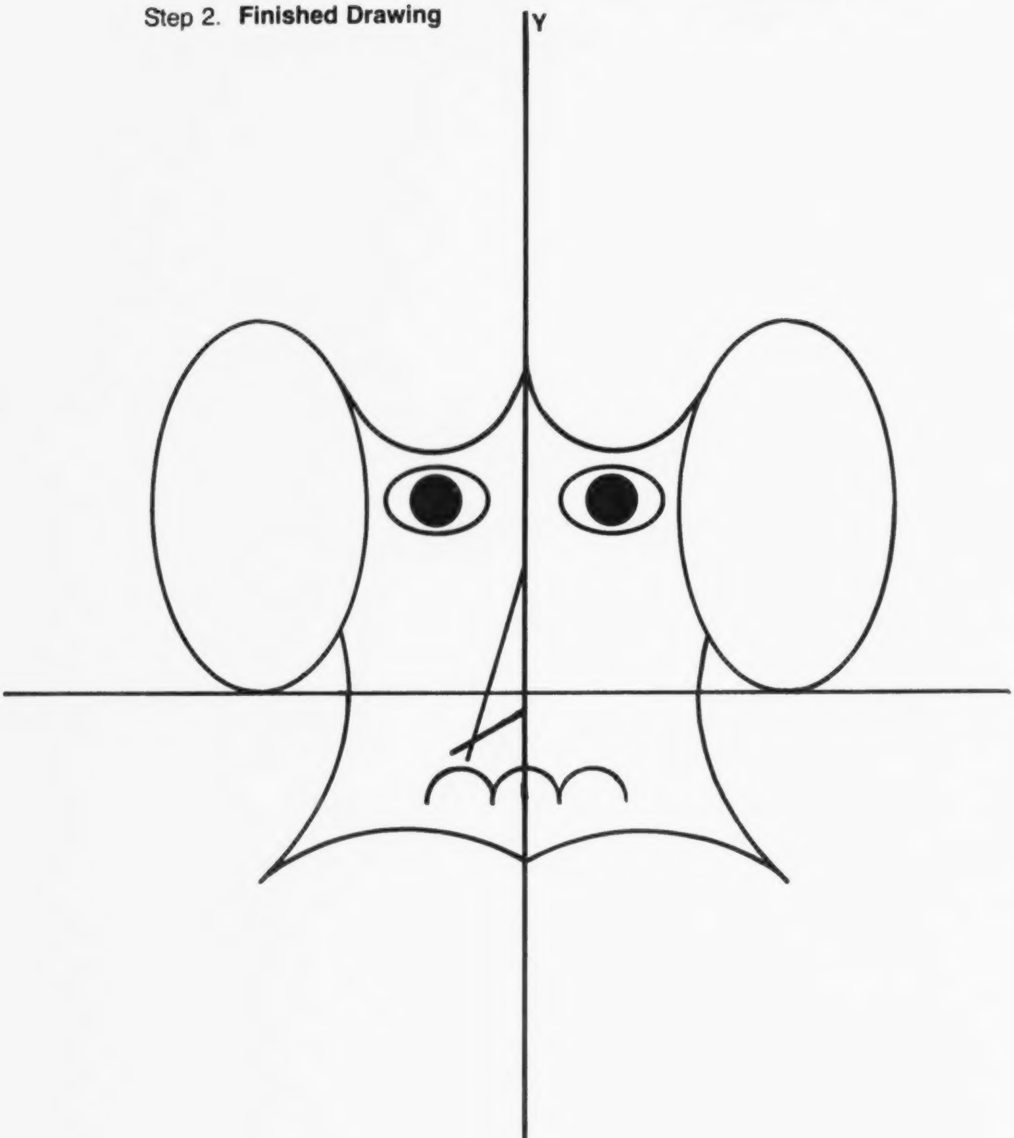
Step 1. Rough Sketch



Solution:

Example 1

Step 2. **Finished Drawing**



Solution:

Step 3.

Part of Figure	Conic Section or Portion	Expressed in Terms of X and Y
Left ear	Ellipse	$\frac{(X-{}^{-}6)^2}{16}+\frac{(Y-4)^2}{4}=1$
Right ear	Ellipse	$\frac{(X-6)^2}{16}+\frac{(Y-4)^2}{4}=1$
Left eye outline	Ellipse	$\frac{(X-{}^{-}2)^2}{4}+\frac{(Y-4)^2}{1}=1$
Right eye outline	Ellipse	$\frac{(X-2)^2}{4}+\frac{(Y-4)^2}{1}=1$
Left eye iris	Circle and part shaded	$(X-{}^{-}2)^2+(Y-4)^2\leq 4/9$
Right eye iris	Circle	$(X-2)^2+(Y-4)^2\leq 4/9$
Jaws	Hyperbola	$\left\{(X,Y)\ \frac{X^2}{9}-\frac{Y^2}{9}=1,\ -7\leq Y\leq 4\right\}$
Nose	Two intersecting lines	$Y=3X+2\quad Y-X=0-1\leq X\leq 0$
Top of head–left	Arc of circle	$\{(X,Y):(X-{}^{-}3)^2+(Y-12)^2=25,\ {}^{-}5\leq X\leq 0, 6\leq Y\leq 7\}$
Top of head–right	Arc of circle	$\{(X,Y):(X-3)^2+(Y-12)^2=25,\ 0\leq X\leq 5, 6\leq Y\leq 7\}$
Bottom of head–left	Arc of circle	$\{(X,Y):(X-{}^{-}3)^2+(Y-9)^2=25, -5\leq Y\leq -4\}$
Bottom of head–right	Arc of circle	$\{(X,Y):(X_{-}3)^2+(Y-{}^{-}9)^2=25, -5\leq Y\leq -4\}$
Left part of mouth	Arc of circle	$\{(X,Y):(X-{}^{-}2)^2+\ (Y-{}^{-}7/2)^2=1, -3\leq X\leq -5/2\}$
Center part of mouth	Arc of circle	$\{(X,Y):X^2+(Y-{}^{-}7/2)^2=1, -5/2\leq X\leq +5/2\}$
Right part of mouth	Arc of circle	$\{(X,Y):(X-2)^2+\ Y-{}^{-}7/2)^2=1, -5/2\leq X\leq 3\}$

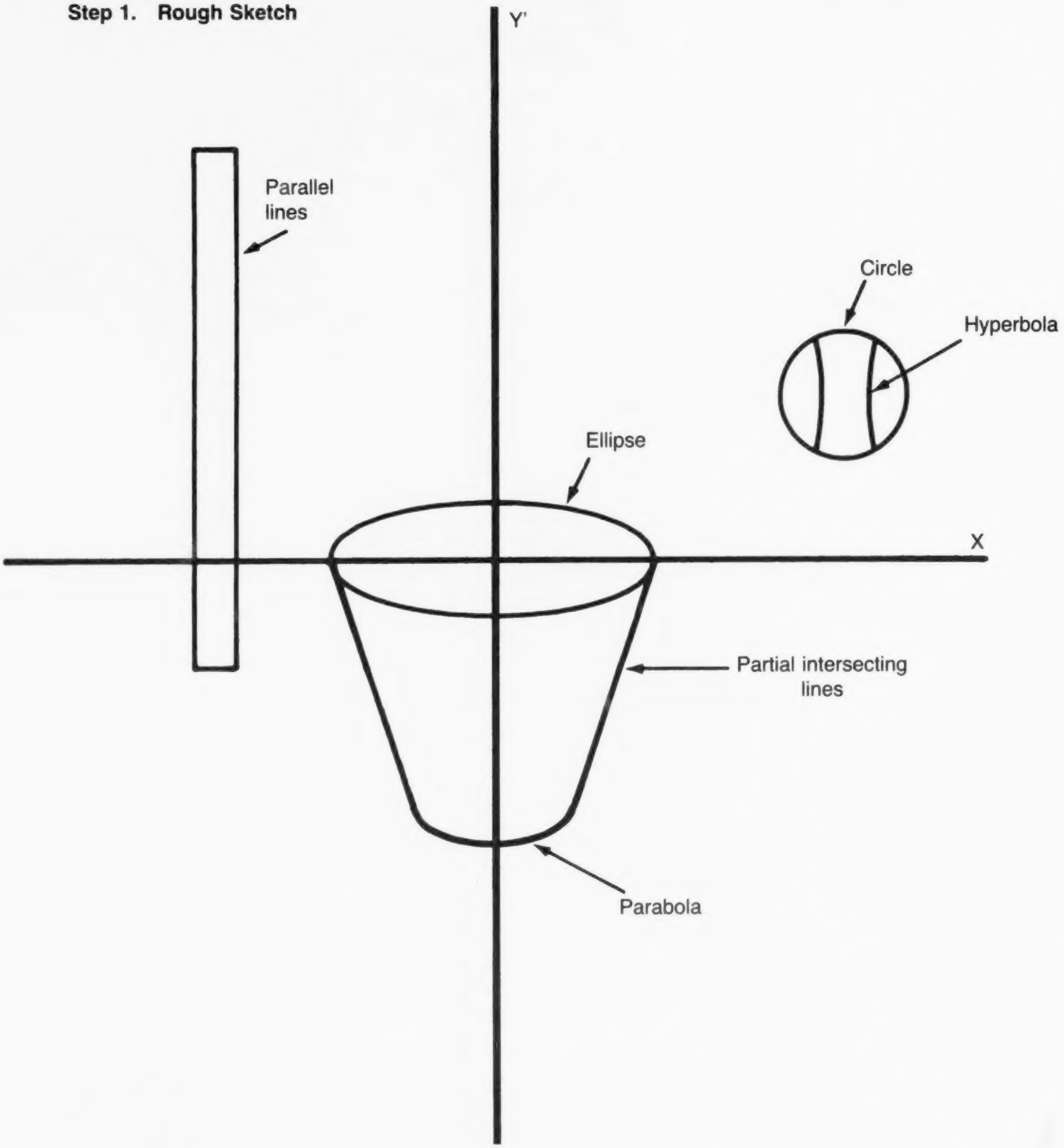
Step 4. Name is shown on graph.

Note: In making the rough sketch and transferring it to the Cartesian coordinate plane, some of the equations you anticipate to fit conic sections in your rough drawing have to be changed in the actual drawing. An example of this is in the rough sketch I anticipated semicircles for the top of the head, but they did not turn out as semicircles.

Solution:

Example 2

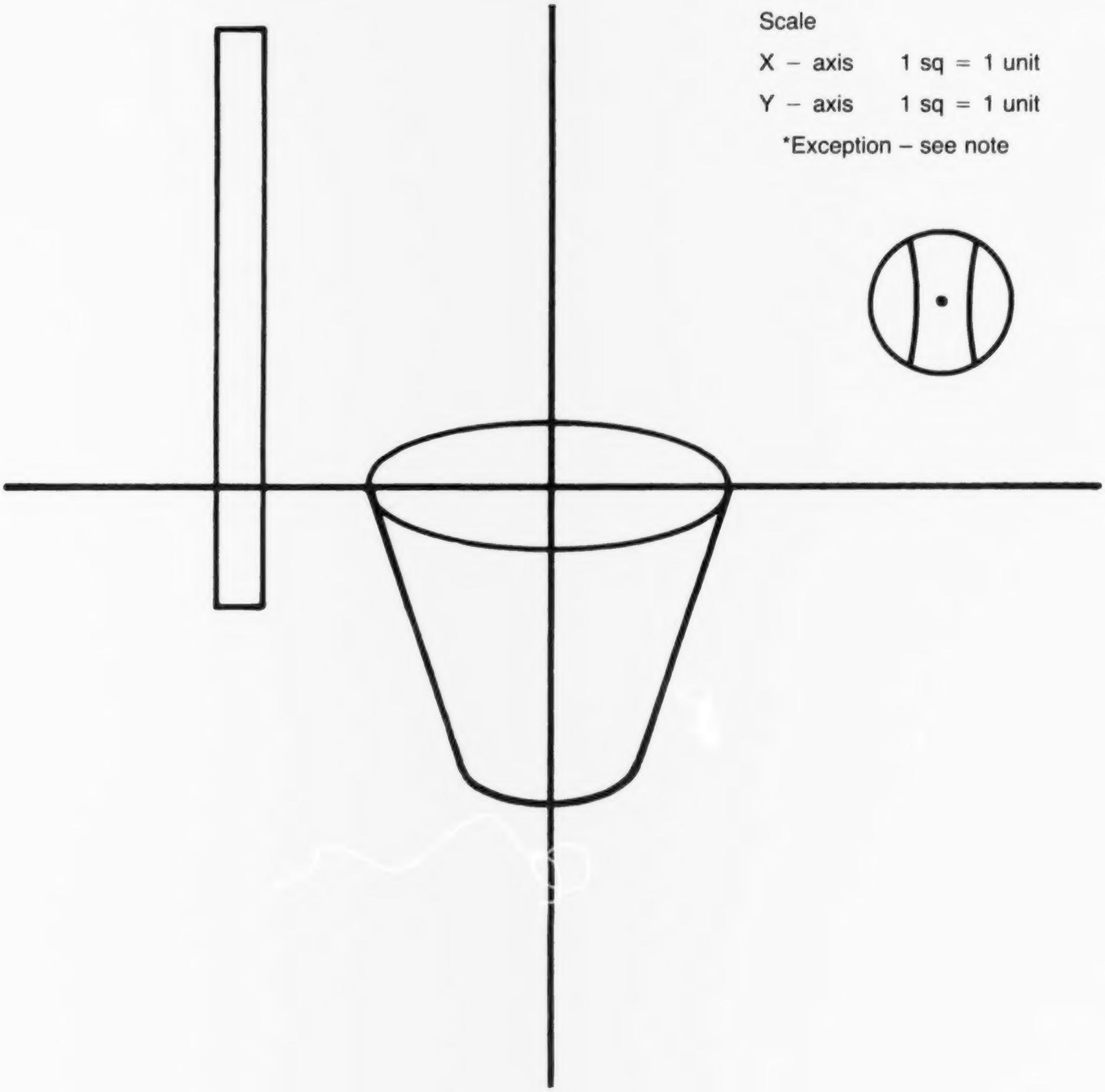
Step 1. Rough Sketch



Solution:

Example 2

Step 2. Finished Drawing



Scale

X – axis 1 sq = 1 unit

Y – axis 1 sq = 1 unit

*Exception – see note

SLAM BAM

Solution:

Step 3.

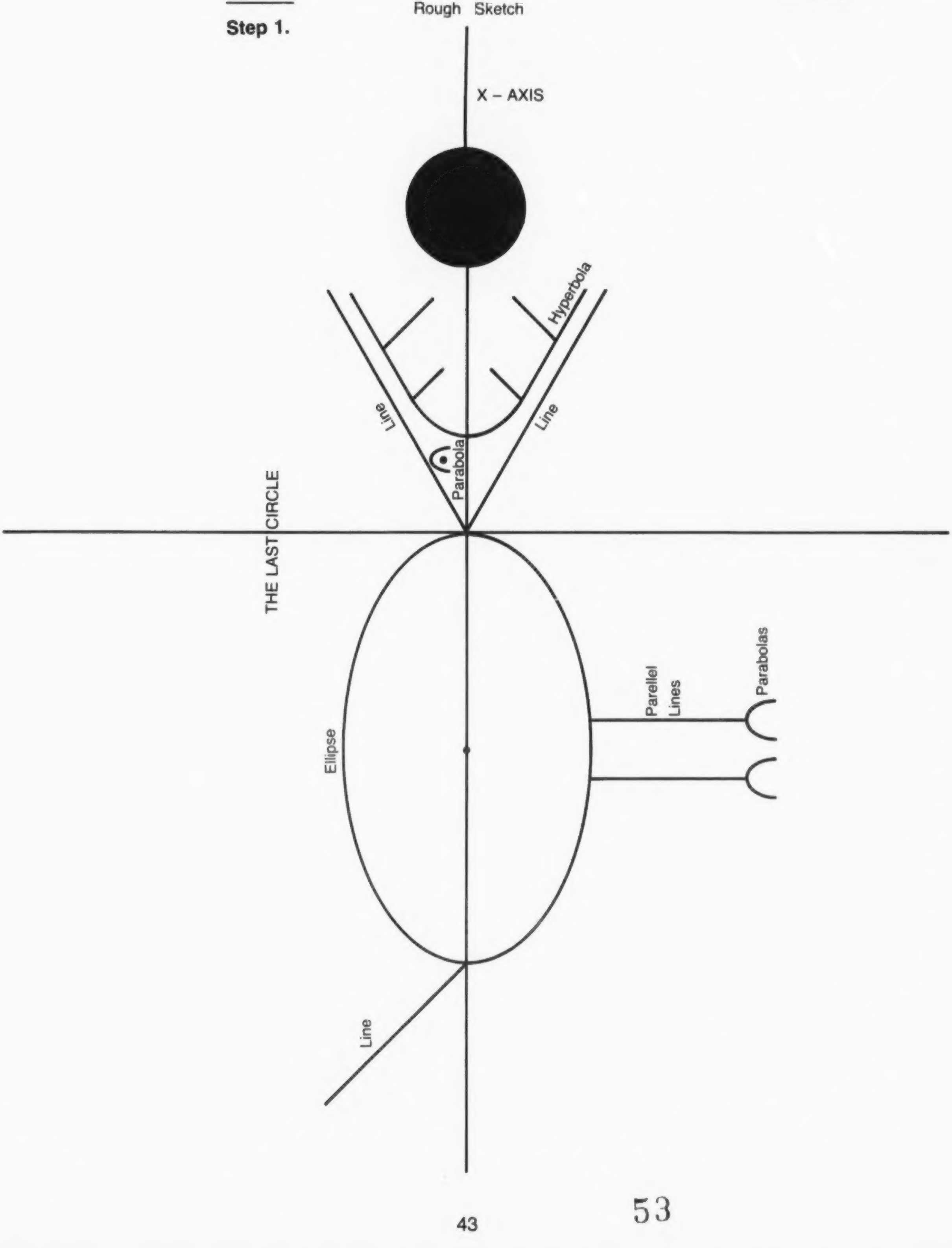
Part of Figure	Conic Section or Portion	Expressed in Terms of X and Y
Backboard	Line	$Y = 0 \quad D = [X: -7 \leq X \leq -5]$
		$\begin{cases} X = -7 \\ X = -9 \end{cases} R = [Y: -3 \leq Y \leq 30]$
		$\begin{cases} Y = -3 \\ Y = 30 \end{cases} D = [X: -9 \leq X \leq -7]$
Basket	Ellipse	$\frac{X^2}{25} + \frac{Y^2}{9}$
Ball	Circle	$(X - 9)^2 + (Y - 15)^2 = 9$
	Hyperbola	$4 (X - 9)^2 - (Y - 15)^2 = 4$
Bottom of Basket (net)	Parabola	$X^2 = \frac{9}{2} (Y + 20) \quad D = [X: -3 \leq X \leq 3]$
Net Outline	Line	$Y - 9X + 45 = 0$
		$D = [X: 3 \leq X \leq 5] \quad R = [Y: -18 \leq Y \leq 0]$
		$Y + 9X - 45 = 0$
		$D = [X: -5 \leq X \leq -3] \quad R = [Y: -18 \leq Y \leq 0]$

Note: In the drawing of ball using the circle and hyperbola and scale will be the same for both axis. Scale: 1 square = 2 units.

Solution:

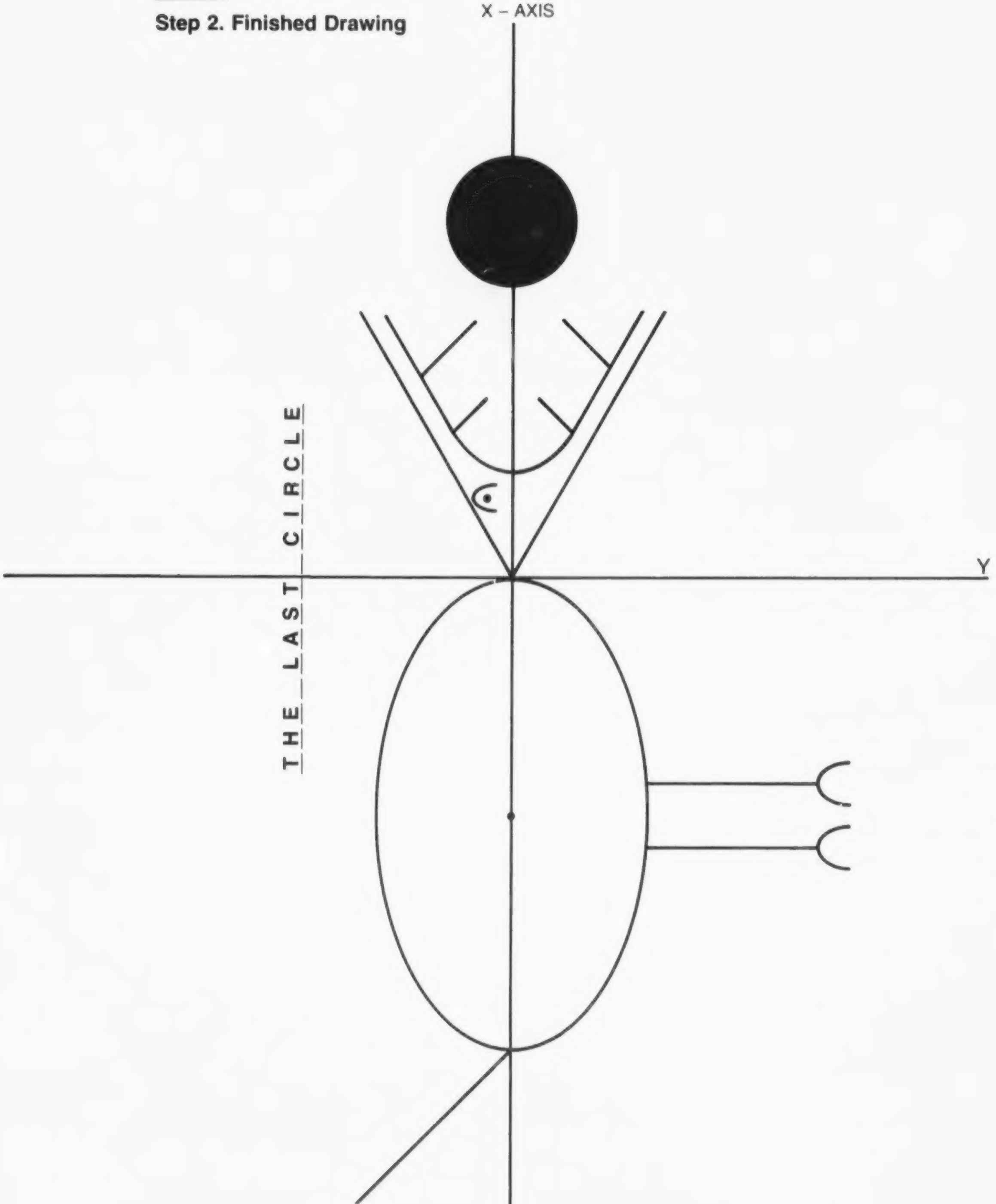
Step 1.

Example 3



Solution:

Step 2. Finished Drawing



Solution: Step 3.

Part of Figure	Conic Section or Portion	Expressed in Terms of X and Y
The Tail	Line	$Y = -(X+6) \quad D = [X \mid -8 \leq X \leq -6]$
The Body	Ellipse	$\frac{(X+3)^2}{9} + \frac{Y^2}{1} = 1$
The Legs	Parallel Lines	$X = -\frac{7}{2}$ or $X = -\frac{5}{2} \quad R = [Y \mid -3 \leq Y \leq -\frac{\sqrt{35}}{6}]$
The Feet	Parabolas	$Y+3 = -1(X+\frac{7}{2})^2 \quad Y+3 = -1(X+\frac{5}{2})^2$ $R = [Y \mid -\frac{7}{2} \leq Y \leq -3]$
The Mouth	Hyperbola	$\frac{X^2}{4} - \frac{Y^2}{1} = 1 \quad D = [X \mid 2 \leq X \leq 4]$
The Head	Intersecting lines	$Y = \pm \frac{1}{2} X \quad D = [X \mid 0 \leq X \leq 4]$
The Eye	Parabola	$(Y - \frac{3}{4}) = -1 \quad (X - \frac{3}{2})^2 \quad R = [Y \mid \frac{1}{4} \leq Y \leq \frac{3}{4}]$
The Eyeball	Point	$(\frac{3}{2}, \frac{1}{2})$
The Food	Circle	$(X-7)^2 + Y^2 \leq 4$
The Teeth	Lines	$Y = \pm(X - \frac{13}{4}), \quad Y = \pm(X - \frac{14+\sqrt{33}}{4})$ $D = [X \mid \frac{5}{2} \leq X \leq 3] \quad D = [X \mid \frac{7}{2} \leq X \leq 4]$

EXTENDED LEARNING: The same activities could include trigonometric functions.

CONSTRUCTING A REGULAR 17-GON

GRADE LEVEL/ COURSE: Geometry

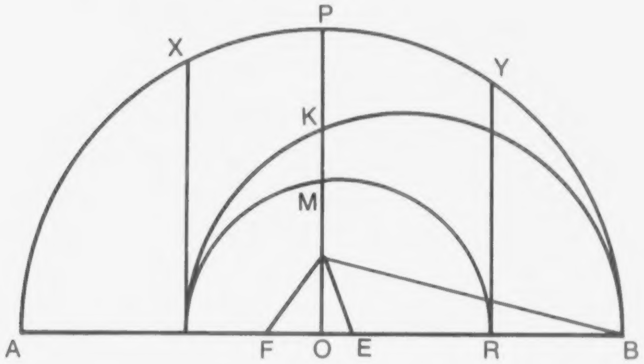
MATERIALS: Compass, straightedge and paper.

INTRODUCTION/ OBJECTIVES: The student will construct a regular 17-gon using a compass and a ruler following the technique developed by H. M. Richmond.

PROCEDURE: With the restriction that one use only a compass and a straight edge, certain regular polygons can be constructed. Polygons with 3, 4, 5, and 6 sides were constructed by Euclid. Polygons with sides that number even multiples of 3, 4, 5, or 6 could easily be constructed from the originals. In 1796 Gauss identified the remaining polygons that could be constructed with ruler and compass. He proved that a regular n-gon could be constructed in this manner only if n were a Fermat prime or a product of several such prime numbers without repetition. The only known Fermat primes are 3, 5, 17, 257, and 65,537. Of these, all have been constructed except the 65,537-gon.

The technique described here for constructing the regular 17-gon was formulated and proved by H. M. Richmond in 1893.

ACTIVITY: Construct a regular 17-gon.



Step 1. Construct a semicircle $\overset{\frown}{AB}$ with center 0.

Step 2. Construct $\overline{OP}$ perpendicular to $\overline{AB}$ with P on $\overset{\frown}{AB}$.

Step 3. Locate point M on $\overline{OP}$ ¼ of the distance from 0 to P.

Step 4. Connect M to B to form $\overline{MB}$.

Step 5. Locate point E on $\overline{OB}$ such that $m\angle OME = \frac{1}{4} m\angle OMB$.

Step 6. Locate point F on $\overline{OA}$ so that $m\angle EMF = 45°$. (This can be done by constructing the perpendicular to $\overline{ME}$ at M and then bisecting the resulting angle.)

Step 7. Locate the midpoint of $\overline{FB}$. Label it R.

Step 8. Using R as a center and $\overline{RB}$ as a radius construct an arc that intersects OP. Label the point of intersection K.

Step 9. Using E as a center and $\overline{EK}$ as a radius construct an arc that crosses $\overline{AB}$ twice. Label these two points N_1 and N_2.

Step 10. Construct a perpendicular to $\overline{AB}$ at N_1 and at N_2. Label the points where these perpendiculars cross $\overset{\frown}{AB}$ as X and Y.

Step 11. The arc $\overset{\frown}{XY}$ has a measure 2/17 of the circle 0. Bisecting this arc will give the proper arc length $\overset{\frown}{XW}$. Complete the construction of the regular 17-gon by marking this length consecutively around circle 0.

REFERENCES:

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Coxeter, H. M. S. *Introduction to Geometry.* New York: John Wiley and Sons, Inc., 1961.

CONSTRUCTION OF A SEGMENT OF LENGTH $\sqrt{X}$

GRADE LEVEL/ COURSE: Geometry

MATERIALS: Compass, straightedge and paper.

INTRODUCTION/ OBJECTIVES: The student will construct a segment of length $\sqrt{X}$ units when given a segment with a length of one unit.

It is often difficult for students to relate irrational values to actual physical measurements. Making measurements using approximate values for irrational numbers often seems inaccurate and therefore impractical. In performing this construction the student can see that each irrational number corresponds to an actual physical measurement that is independent from its approximate value.

PROCEDURE: The students will do this activity in class.

ACTIVITY: Construct a segement with a length of $\sqrt{X}$.

Step 1. Draw a line segment of some random length to be used as the unit segment. The length of this segment is one unit.

Step 2. From this unit segment construct a segment of length X units. (Example: If you wish to construct a segment with length $\sqrt{3}$ units, then X will have to measure 3 times as long as the unit segment.)

Step 3. On a construction line construct segments $\overline{AB}$ and $\overline{BC}$ such that B is between A and C, AB = 1 unit and BC = X units.

Step 4. Locate the midpoint of $\overline{AC}$. Label it M.

Step 5. Using M as a center and MA as a radius construct semicircle $\overset{\frown}{AC}$.

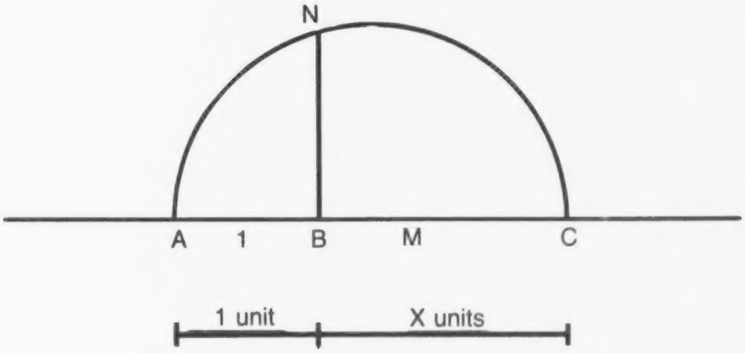
Step 6. Construct the line perpendicular to $\overline{AC}$ at point B. This line will intersect semicircle $\overset{\frown}{AC}$. Label this point of intersection N.

Step 7. The measure of $\overline{BN}$ equals $\sqrt{X}$.

EXTENDED LEARNING:

Proof of the Construction:

Connecting point N to points A and C with straight line segments forms a triangle with segment AC. This triangle, $\triangle ANC$, is a right triangle since it is inscribed in a semicircle. Segment $\overline{BN}$ is the altitude to the hypotenuse for $\triangle ANC$, so it divides $\triangle ANC$ into two small right angles $\triangle ABN$ and $\triangle NBC$. The Angle-Angle Similarity theorem indicates that $\triangle ANC \sim \triangle ABN \sim \triangle NBC$. The sides of these triangles are therefore proportional, i.e., $\frac{BN}{BC} = \frac{AB}{BN}$, and by the law of proportion (BN) (BN) = (BC) (AB). Therefore $(BN)^2 = (X)(1)$ because we designated that BC = X and AB = 1 in step 3. It follows than that BN = $\sqrt{X}$.



REFERENCES: Moise, Edwin. *Elementary Geometry From An Advanced Standpoint.* Reading, Mass.: Addison-Wesley, 1974.

CONSTRUCTIONS AND DISSECTIONS

GRADE LEVEL/ COURSE: Geometry

MATERIALS: Heavy paper (construction or art), straightedge, scissors, tape or glue.

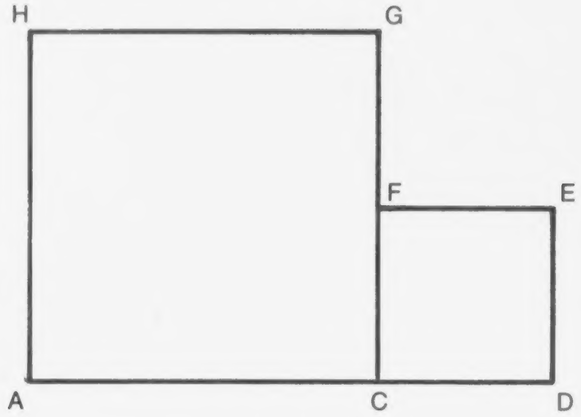
INTRODUCTION/ OBJECTIVES: The student will dissect two squares using accurate construction techniques and form a new square from the resulting pieces.

PROCEDURE:

1. Have the students work individually or in groups of two.
2. Inaccurate constructions will be obvious to the teacher and to the student when the student attempts to form the new square.
3. Have the students construct the larger square so that the sides are no more than half the length of the construction paper.
4. Have the students construct the smaller square adjacent to the larger square with their bases collinear.
5. After cutting, each student should have 5 pieces which can be arranged to form a new square. The new square should be taped or glued to another piece of construction paper.

ACTIVITY 1: **Step 1.** Have students construct two squares, one large and one small, using perpendicular lines. Label the large square A H G C and the small one C F E D. The squares should have their bases collinear and joined at point C. F is between C and G.

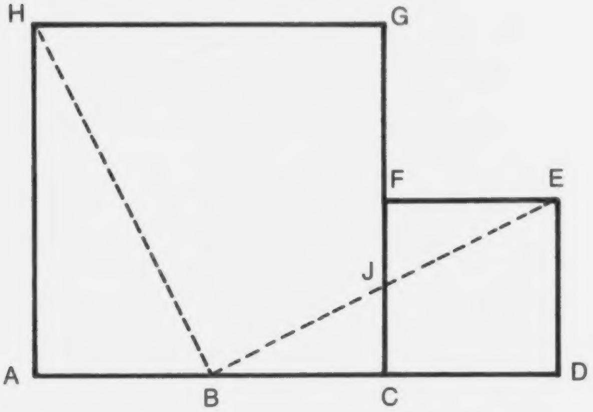
Figure 1



Step 2. Place point B between A and C such that AB = CD.

Step 3. Draw segments $\overline{BH}$ and $\overline{BE}$. Label the intersection of $\overline{CF}$ and $\overline{BE}$ as point J.

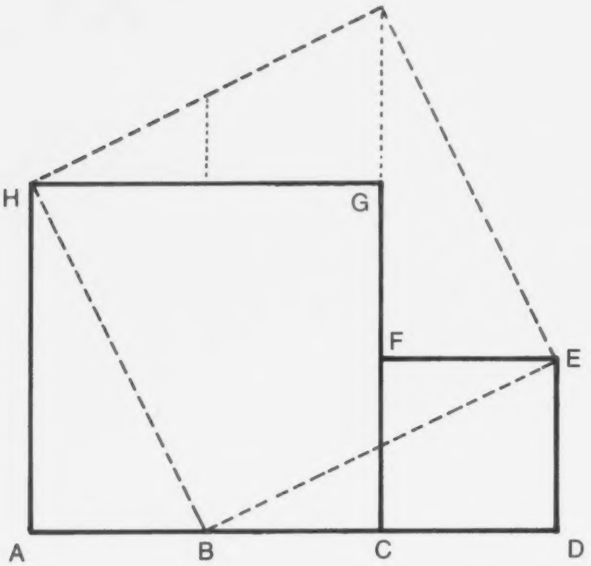
Figure 2



Step 4. Cut out the two squares and make cuts along the segments $\overline{BH}$, $\overline{BJ}$, and $\overline{JE}$. There are now 5 pieces to be arranged to form a square.

Step 5. Have students tape or glue their solutions to paper.

Solution: The segments $\overline{BH}$ and $\overline{BE}$ will be two sides of the new square.

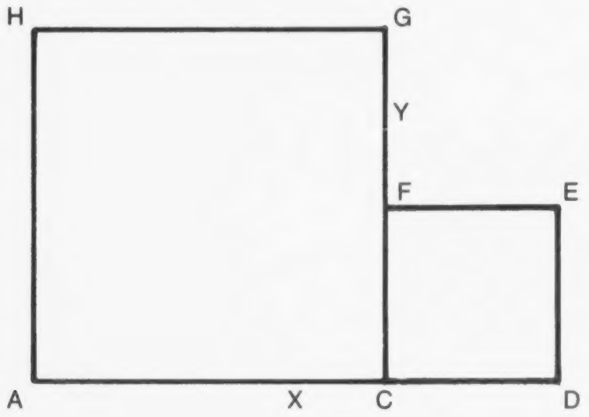


ACTIVITY 2: The following is a more symmetrical solution but less simple than the previous solution.

Step 1. Again have students construct two squares as described in Activity A.

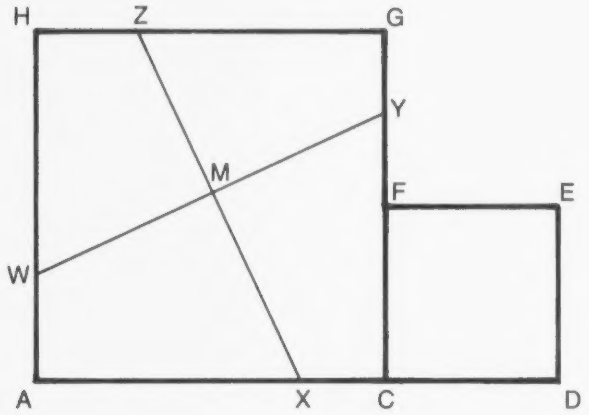
Step 2. Place point X at the midpoint of $\overline{AD}$.

Step 3. Place point Y such that CY = AX.



Step 4. Locate the center M of the larger square by drawing diagonals lightly in pencil.

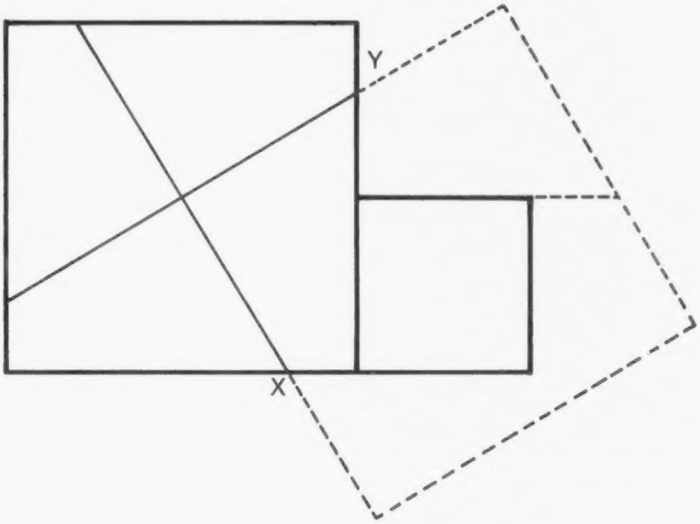
Step 5. Draw a segment from X through M to the opposite side of the larger square. Also draw another segment from Y through M to the opposite side of the larger square. Label the endpoints Z and W respectively.



Step 6. Cut out the two squares and make cuts along $\overline{XZ}$ and $\overline{WY}$.

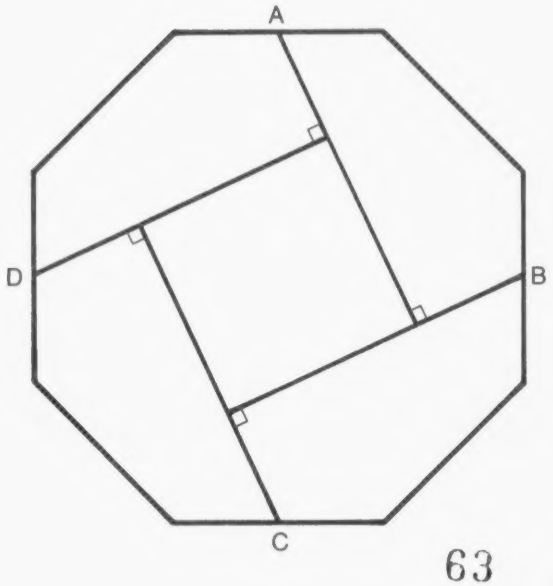
Step 7. The student now has 5 pieces to arrange to form a square.

Solution:



EXTENDED LEARNING:

1. Construct a regular octagon
2. Place points A, B, C, and D as midpoints of every other side.
3. Draw perpendicular intersecting lines that form a square in the center. Note: The lengths of the sides of the square must be equal to the lengths of the sides of the octagon.
4. Cut on the lines drawn to form five pieces.
5. Arrange the pieces to form a square.



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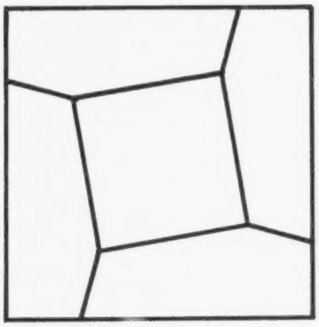
Solution:



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Jacobs, Harold R. *Mathematics: A Human Endeavor.* New York: W. H. Freeman and Company, 1982.

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CURVE STITCHING

GRADE LEVEL/ COURSE: 7–8

MATERIALS: Lightweight paper plates, sewing thread in three contrasting colors, needles, scissors, pencils, rulers, "sample design" instruction sheets.

INTRODUCTION/ OBJECTIVES: The student will investigate designs and properties of uniquely formed curved lines.

PROCEDURE: In order to create interest show students some previously created designs in a variety of patterns and/or mediums. This is designed for a classroom activity which is to be completed at home if time is limited. Highly motivated students could follow the activities listed below to work independently at home.

ACTIVITY:

Step 1. On a paper plate, lightly sketch an equilateral triangle.

Step 2. Label the vertices of the triangle A, B, and C.

Step 3. Mark off 19 equal segments on each side of the triangle.

Step 4. Position the triangle so that $\overline{AB}$ is the base.

Step 5. From the left to right, label the points *between* A and B as follows: 1, 2, 3, . . . 18.

Step 6. Position the triangle so that $\overline{BC}$ is the base and label the points between C and B as in step 5.

Step 7. Position the triangle so that $\overline{CA}$ is the base and label the points between C and A as in step 5.

Step 8. Thread needle with thread of one color and knot end of thread.

Step 9. From underside of plate, pull needle and thread up through point A. Then, from top side of plate, push needle and thread down through point B.

Step 10. From underside of plate, pull needle and thread up through point "1" of side $\overline{AB}$ and push down through point "1" of side $\overline{BC}$. (Needle points should be made slightly inside string lines $\overline{AB}$ and $\overline{BC}$.

Step 11. Repeat process described in steps 9 and 10 until all points on $\overline{AB}$ are stitched to corresponding points on $\overline{BC}$.

Step 12. With the last thread, repeat steps 9–11 using the points on BC and CA.

Step 13. With the last thread, repeat steps 9–11 using the points on CA and AB.

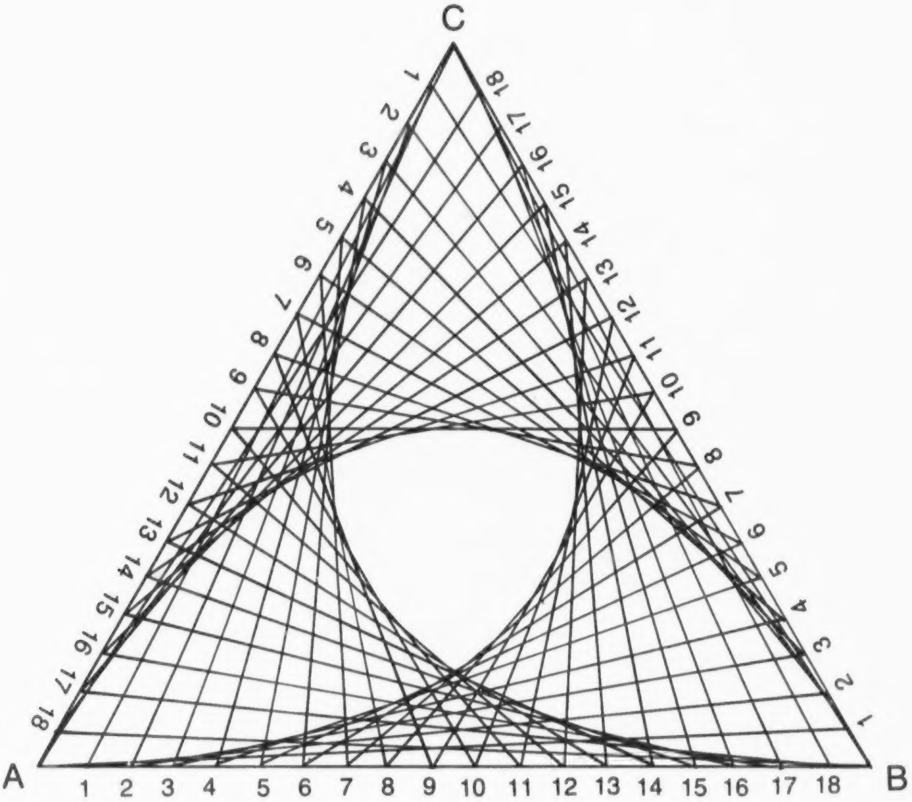


DIAGRAM FOR SAMPLE DESIGN

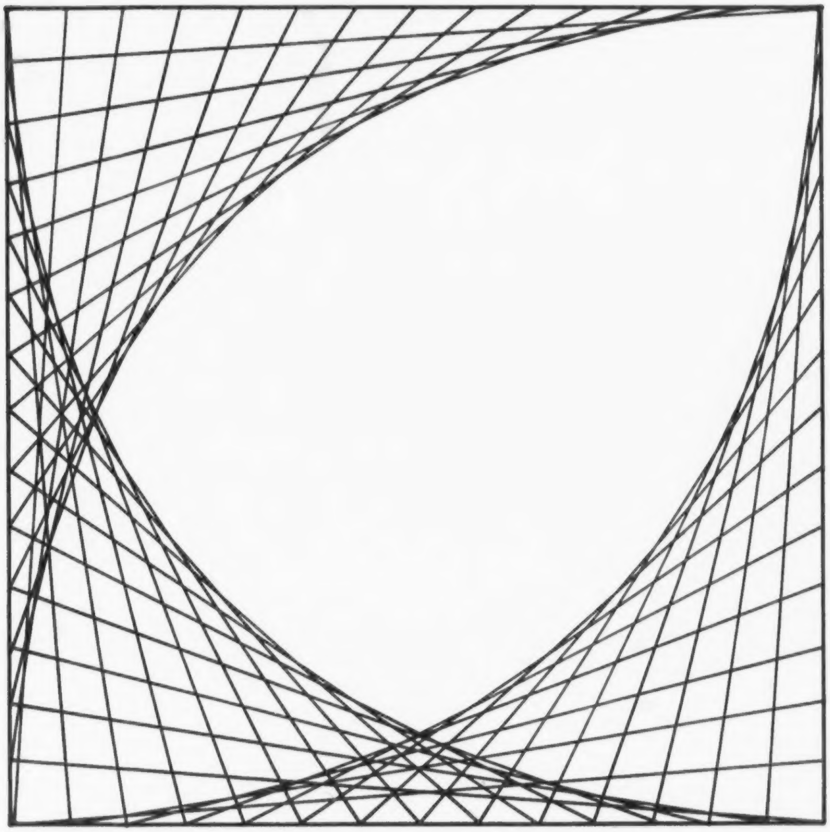
A. By analyzing his completed design, the student will observe the following truths:

1. Straight lines can create an illusion of curves.
2. Any design formed by one color has symmetry with respect to a line from its vertex perpendicular to the opposite side of the triangle.
3. The composite design creates a closed curve within the triangle.
4. Individual string lines intersect to form irregular quadrilaterals.

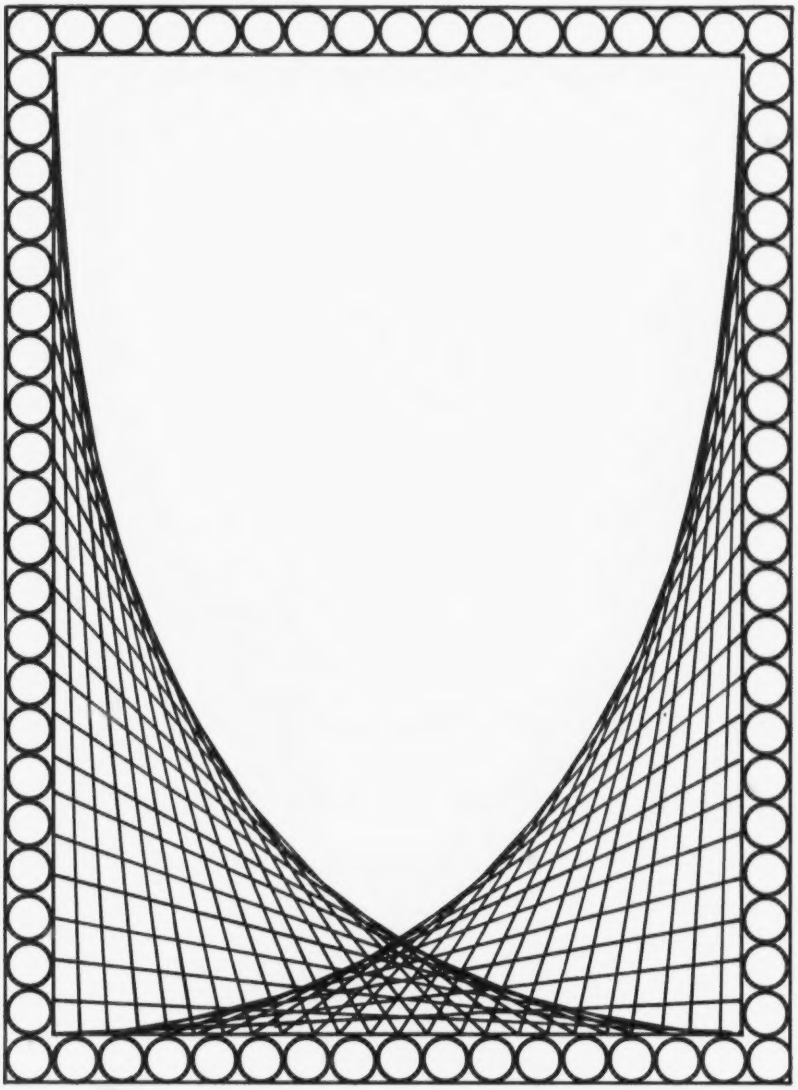
B. In addition, Algebra II students may investigate the following truths:

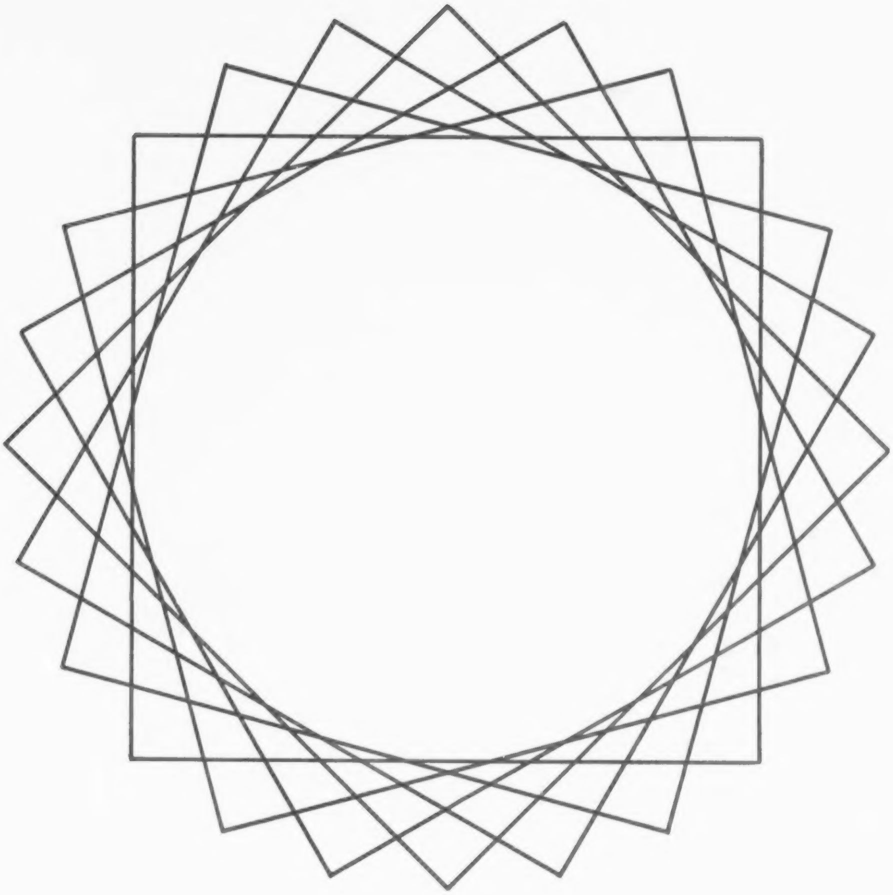
1. The curve formed by the strings of one color is a parabola.
2. The focus of a parabola lies on a line joining the center of the triangle and a vertex of the triangle.

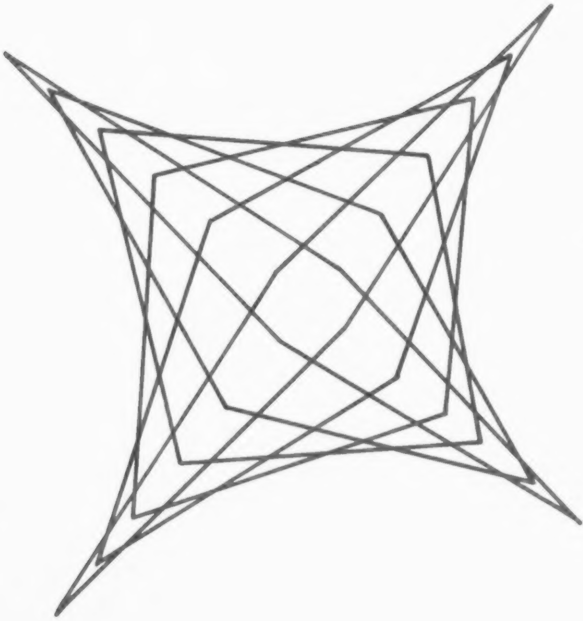
EXTENDED LEARNING: 1. Students at any level will create curve stitched designs based on geometric figures other than equilateral triangles (see samples below).

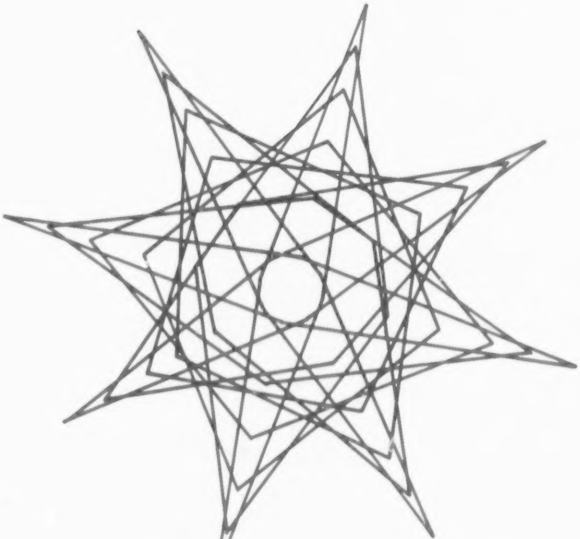


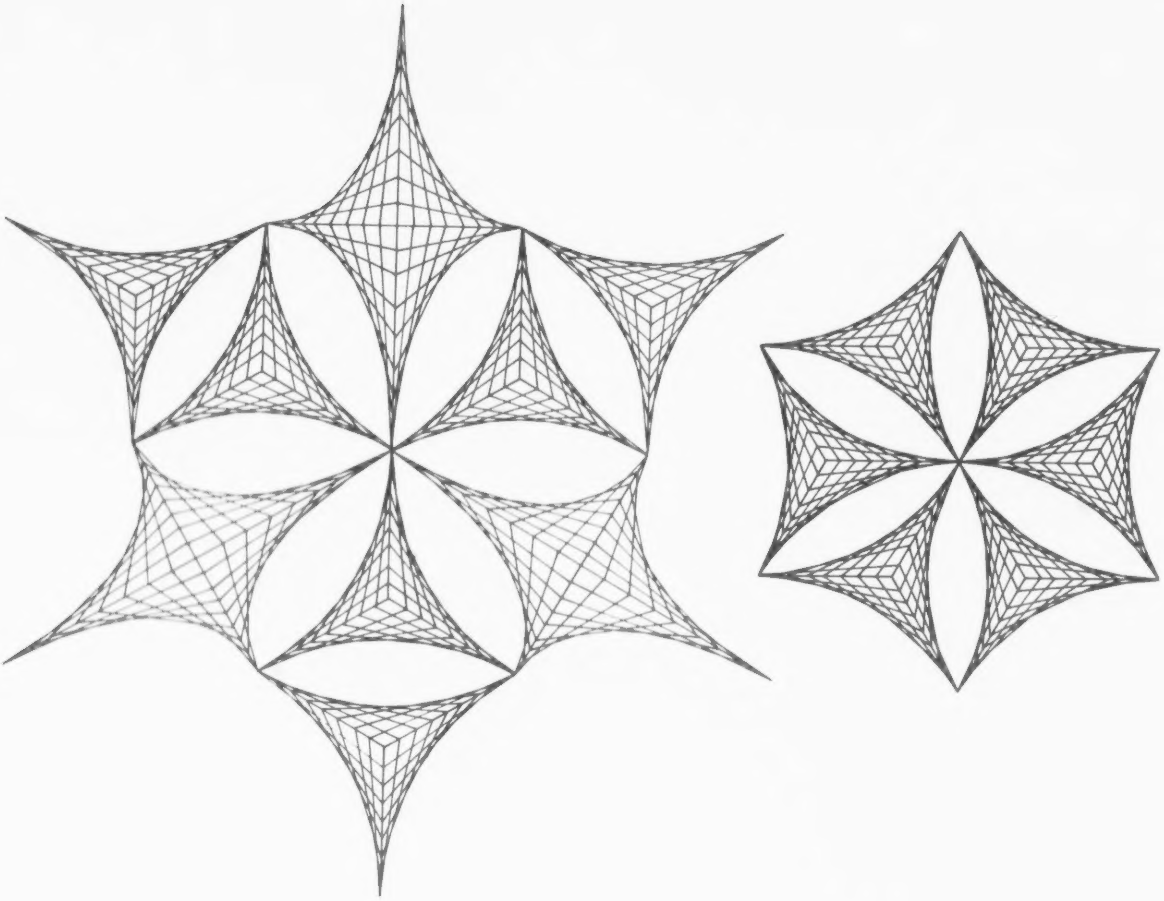
SAMPLE DESIGNS OF CURVE STITCHING











2. Students at any level will curve stitch designs using various mediums.
   - A. Heavyweight paper plates with embroidery thread.
   - B. Poster board with yarn.
   - C. Board and nails with string.
   - D. Paper or poster board with ink or pencil.

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Posamentier, Alfred S., and Stephelman, Jay. *Teaching Secondary School Mathematics; Techniques and Enrichment Units.* Columbus, Ohio: Merrill Publications, 1986.

Menninger, K. W. *Mathematics in Your World.* New York: The Viking Press, 1961.

Seymour, Dale and Schadler, Reuben. *Creative Constructions.* Palo Alto, California: Creative Publications, 1974.

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DEVELOPING A PROBLEM SOLVING STRATEGY

GRADE LEVEL/ COURSE: 2–4

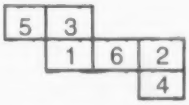
MATERIALS: Puzzle forms (see examples)

INTRODUCTION/ OBJECTIVES: The student will test conjectures for solving the puzzle in order to gain experience in a problem solving technique (guess and test) and eventually develop a strategy.

PROCEDURE: This activity is appropriate for individuals, or small groups (2–4 students). Students will need to know the definition of *consecutive numbers*.

ACTIVITY: Have students begin with the blank cell-arrangement below and arrange the numbers shown in the spaces, so that consecutive numbers are *not* in spaces that "touch" on a side *or* on corner.

Example:



3 touches 5, 1, 6
6 touches 1, 3, 2, 4
6 does not touch 5, etc.
But no two consecutive numbers "touch"!

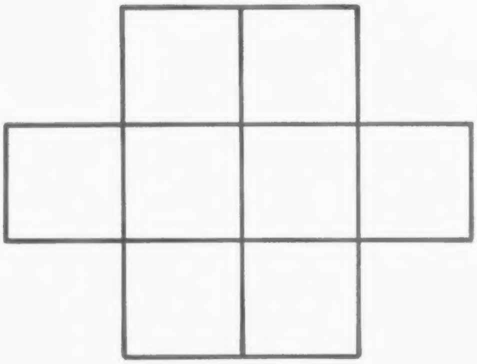
Perhaps after a first attempt (and an eventual successful effort), students should be encouraged to begin to verbalize a possible strategy. This may not become clear until other attempts, with arrangements as shown below have been tried. Each successive attempt should bring students even closer to verbalizing something such as:

"Each first and last number has only one "consecutive number" to be concerned with, and therefore should be placed in the cells which touch the most other cells — the middle cells.

Then . . ."

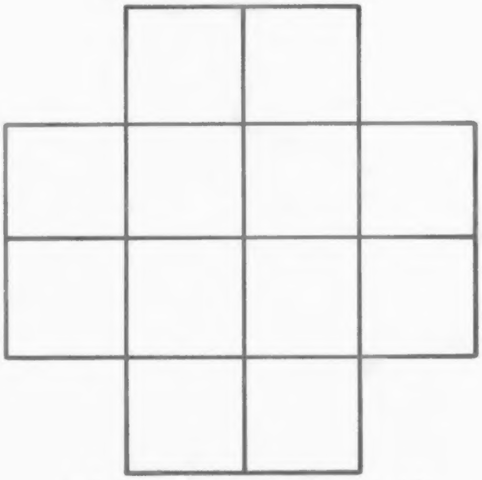
Students should be encouraged to develop individual strategies, as there need not necessarily be just one correct scheme. Use the following arrangements to continue sharpening this problem solving skill, and then, if necessary, build additional (teacher created) arrangements.

FOUR-CROSS



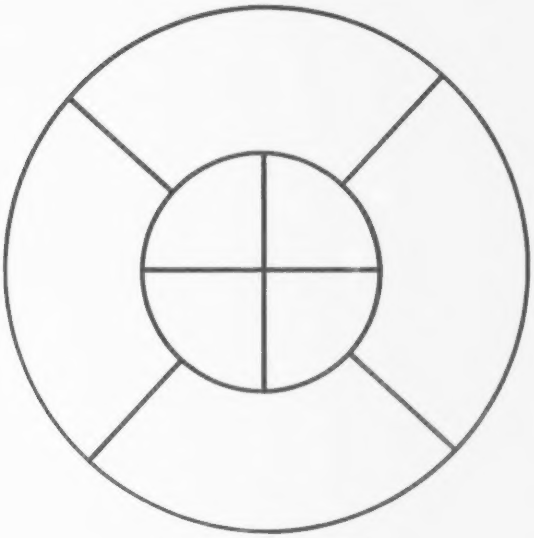
1, 2, 3, 4, 5, 6, 7, 8

DOUBLE FOUR-CROSS



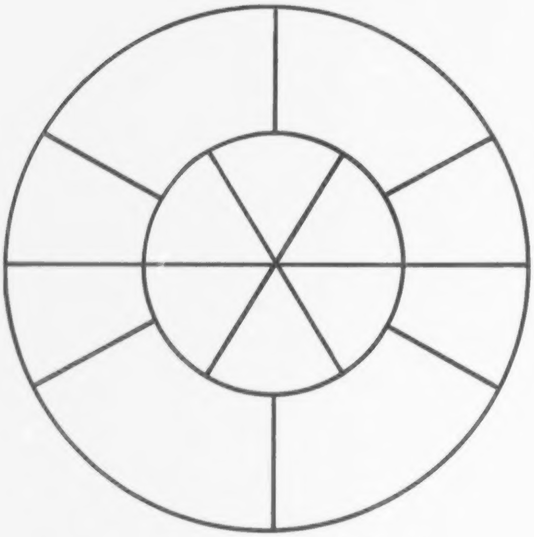
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

WAGON-WHEEL



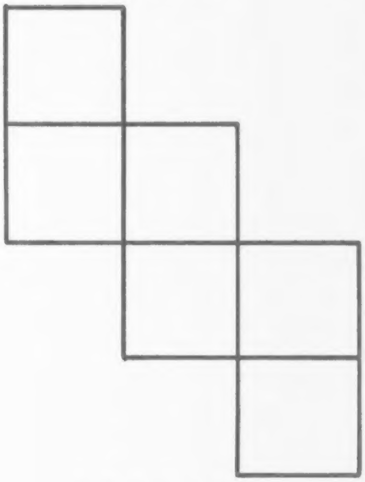
1, 2, 3, 4, 5, 6, 7, 8

DOUBLE WAGON-WHEEL



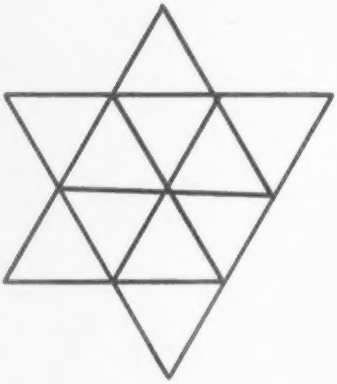
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
12, 13, 14

STAIR STEP



1, 2, 3, 4, 5, 6

STAR SHIP



1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11

EXTENDED LEARNING: Students will design their own puzzles.

DRAWING WITH RECTANGULAR COORDINATES

GRADE LEVEL/ COURSE: 4–6

MATERIALS: Each student will need 18 in. by 24 in. drawng paper, a ruler, and a grid picture which has been duplicated. (See next page.)

INTRODUCTION/ OBJECTIVES: This activity offers an opportunity for students who have mastered rectangular coordinates to apply the concept as they enlarge a drawing by increasing the space between coordinates and copying the lines in each segment of a grid picture.

PROCEDURE: This activity is best accomplished with the teacher directing the large group and giving a demonstration of the process.

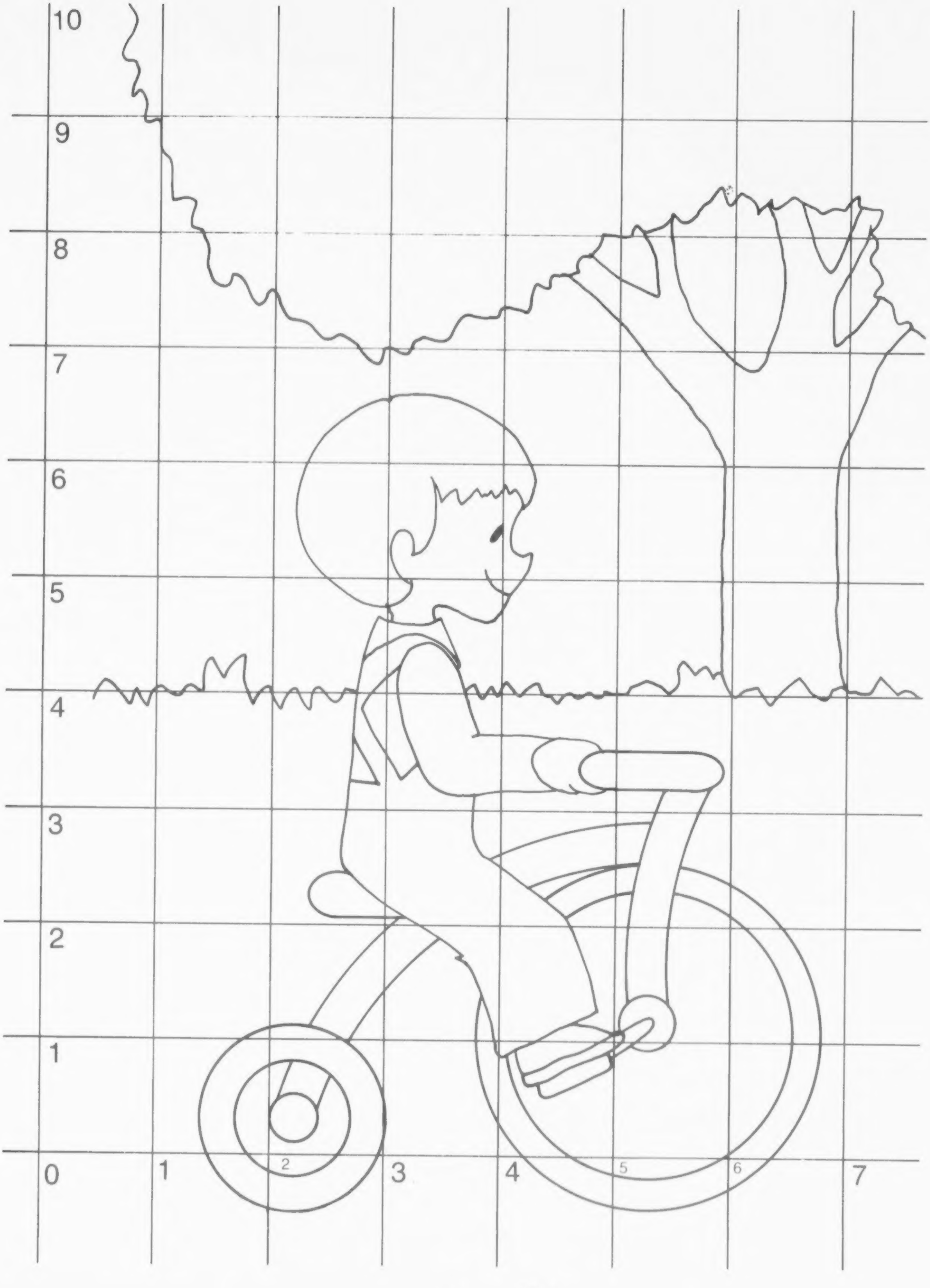
ACTIVITY 1: The teacher will cover the following points in discussion/demonstration.

Step 1. Look at the coordinates on the grid picture.

Step 2. Draw the coordinates on the 18 in. by 24 in. drawing paper so that you will be able to enlarge the picture.

Step 3. Now, enlarge the drawing by copying the lines in each section of the grid picture on to the corresponding section of the grid you have drawn.

ACTIVITY 2: Select a picture from a coloring book and enlarge it using the same method.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EGYPTIAN ARITHMETIC — G RULE

GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVES: This activity will expose the student to the Egyptian system of expressing fractional amounts in order to compare and determine a relationship between this system and the present Hindu Arabic system.

PROCEDURE:

1. The teacher should present a brief historical overview concerning the Egyptians and their contributions in the areas of mathematics and science.
2. The teacher should guide the students to understand the G rule, a method used by the Egyptians to add and subtract unit.
3. The teacher should encourage the students to make generalizations concerning the two number systems.

HISTORY: The Ancient Egyptians' system of arithmetic expressed all fractions as unit fractions (numerator of one). Those with numerators larger than one were expressed as the sums of distinct unit fractions, so that $3/4$ would be $1/2 + 1/4$. The only exception is $2/3$ which had its own symbol.

Since numerators were no larger than 1, only a few fractions could actually be combined by addition. Only those fractions whose denominators were even multiples of each other could be added. This was done with a method called the G rule.

ACTIVITY: For pairs of fractions where one denominator is twice the other, the G rule states that they can be added only if the larger denominator can be divided by 3 and this division gives the denominator of the sum. *Examples:* $1/3 + 1/6 = 1/2$.

Have the students try the following fractions to see which can be added using this rule. Answers are in parentheses.

1. $1/3 + 1/6$ ($1/2$)
2. $1/6 + 1/12$ ($1/4$)
3. $1/7 + 1/14$
4. $1/9 + 1/18$ ($1/6$)
5. $1/8 + 1/16$
6. $1/12 + 1/24$ ($1/8$)
7. $1/18 + 1/36$ ($1/12$)

When one denominator is a different multiple of the other, a more generalized version of the G rule must be used. This form states that if one denominator is n times another, the denominator of the sum is found by dividing the larger denominator by (n + 1). This quotient must be a whole number or the fractions cannot be added. Any fractions that did not fit this pattern could not be combined by the Egyptians. What is the largest answer that can be found? ($1/2$)

Have the students see which of these fractions can be added and which do not give a sum which is a unit fraction. Answers are in parentheses.

$1/2 + 1/12$ ($1/3$)
$1/8 + 1/24$ ($1/6$)
$1/12 + 1/60$ ($1/10$)
$1/2 + 1/6$
$1/5 + 1/15$
$1/12 + 1/36$ ($1/9$)
$1/10 + 1/40$ ($1/8$)
$1/8 + 1/16$
$1/2 + 1/12$
$1/5 + 1/20$ ($1/4$)
$1/6 + 1/30$ ($1/5$)
$1/4 + 1/8$

Let's look at a further extension of the G rule which involves subtraction.

($1/2 - 1/6 = 1/3$; $1/4 - 1/12 = 1/6$; $1/6 - 1/18 = 1/9$)

Have the students notice that with one simple alteration, the G rule applies to the subtraction of unit fractions. Thus, for the first equality, one says 2 into 6 is 3 $(3 - 1)$ is 2, and 2 into 6 is 3 so that the difference of the unit fractions $\frac{1}{2}$ and $\frac{1}{6}$ is $\frac{1}{3}$.

Have the students produce other examples, then have them attempt to state the extension of the subtraction version of the G rule, which is, if one of two unit fractions is n times the other, then their difference is found by dividing $(n - 1)$ into the larger number providing the result is a whole number. Have the students discuss the limitations of this rule.

Have the students find the sum and the difference of the following using the G rule and its extensions:

$\frac{1}{20} \quad \frac{1}{60} =$; $\quad \frac{1}{5} \quad \frac{1}{20} =$; $\quad \frac{1}{7} \quad \frac{1}{42} =$;

$\frac{1}{14} \quad \frac{1}{84} =$; $\quad \frac{1}{15} \quad \frac{1}{30} =$; $\quad \frac{1}{9} \quad \frac{1}{18} =$;

Note: Several can only be added.

REFERENCE:

Gilling, Richard J. *Mathematics in the Time of the Pharaohs.* Mineola, New York: Dover, Publications, Inc., 1972.

Posamentier, Alfred S. and Stepelman, Jay. *Teaching Secondary School Mathematics.* Columbus, Ohio: Merrill Publications, 1981.

Zaslavshy, Claudia. *Africa Counts.* Westport, Connecticut: Lawrence Hill Publications, 1973.

EVEN STEVEN
ODDS AND EVEN

GRADE LEVEL/ COURSE: 3

MATERIAL: Prepared models for odd and even numbers 2–9 (see attached samples)

INTRODUCTION/ OBJECTIVES: When students are familiar with odd and even numbers, they will extend and apply their knowledge by using the concepts of odd and even to solve problems.

PROCEDURE: This exploration would be done best with a small group.

ACTIVITY:

Step 1. Provide each student with models. Have them discuss the configurations for odd and even numbers.

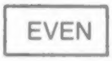
Step 2. Combine various models of even numbers and make a statement about the sum of even numbers.

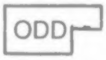
Step 3. Combine various models of odd numbers and make a statement about the sum of odd numbers.

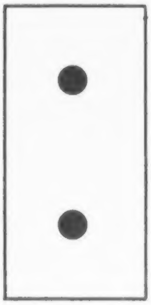
Step 4. Combine various models of odd and even numbers and make a statement about the sum of odd and even numbers.

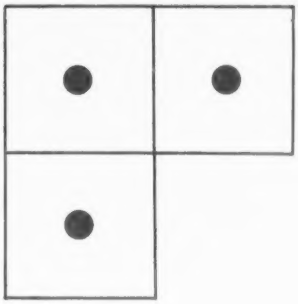
Step 5. Have the students summarize what they know about:
even numbers added to even numbers
even numbers added to odd numbers
odd numbers added to odd numbers

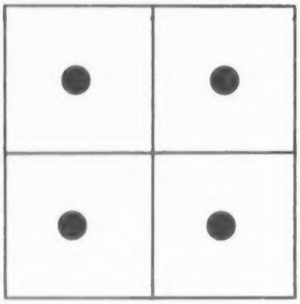
EXTENDED LEARNING: Explore multiplication of even and odd numbers. Use models without the lines and dots.

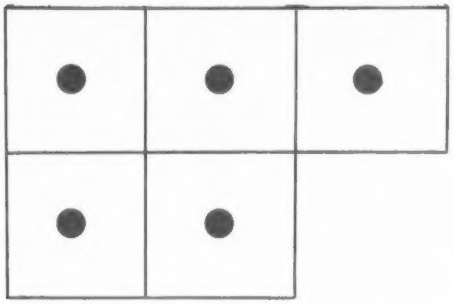


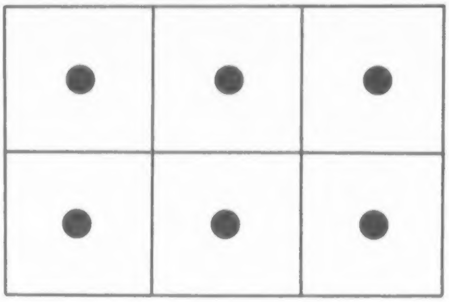


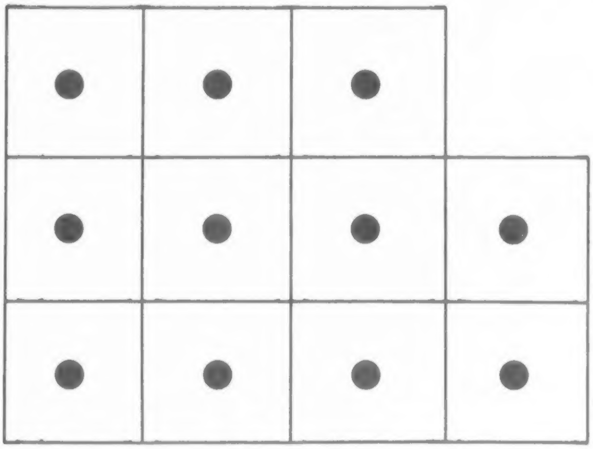


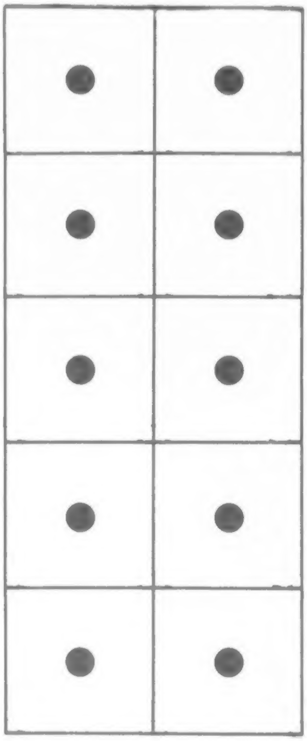


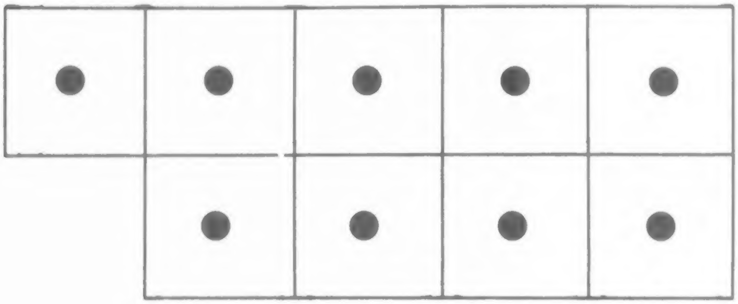












EXPERIMENTS IN SYMMETRY

GRADE LEVEL/ COURSE: 4–5

INTRODUCTION/ OBJECTIVES: The student will perform experiments in symmetry and create symmetrical designs for stimulating analytical and divergent thinking.

PROCEDURE: This activity offers students who have mastered the concept of symmetry with one axis an opportunity to expand their understanding. The experience is intended for individuals to begin independently as they develop the skills required to accomplish the task. The teacher will need to provide reference material on symmetry or assist the students in locating material in the 510 section of the library.

ACTIVITY: The following directions should be given to the students:

Step 1. Research the terms listed below. Following your research, write a definition for each term:

axes of symmetry

bilateral symmetry

Step 2. Experiment with cutting letters of the alphabet from folded paper. How many can be cut so they have vertical or horizontal symmetry? Do any have bilateral symmetry?

Step 3. Using the art materials available in your class room, create a design which has 3 axes of symmetry. List as many objects, man-made or natural as you can find with 3 axes of symmetry.

EXTENDED LEARNING: Do the above activity with 4 axes of symmetry.

REFERENCE: Razzell, Arthur G. *Symmetry*. Doubleday and Company, 1968.

EXPONENTS AND RADICALS

GRADE LEVEL/ COURSE: Algebra I

INTRODUCTION/ OBJECTIVES: Students have already completed the basic principles of exponents and radicals. The problem is used as an enrichment assignment.

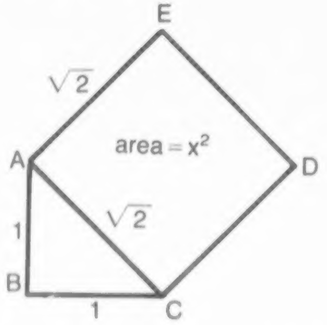
Given the value of $x = \sqrt{2}$, the students will be able to find out the values of x^2, x^4, and x^6 and draw the geometric figures related to each area in algebraic terms.

PROCEDURE: If $x = \sqrt{2}$ find the geometric figures with areas x^2, x^4, and x^6. One must first construct a segment whose length is $\sqrt{2}$. This is done by drawing an isosceles right triangle with vertices A, B and C so that AB = BC = 1 unit. By the Pythagorean formula the hypotenuse AC has a measure of $\sqrt{2}$ units. Using this triangle ABC as a base figure create squares or rectangles with areas measures $(\sqrt{2})^2$, $(\sqrt{2})^4$, and $(\sqrt{2})^6$. The following activities illustrate the procedure used in each case.

ACTIVITY: **Step 1. For the area of x^2**

(a) x^2 is $x \cdot x$; therefore $x^2 = \sqrt{2} \cdot \sqrt{2}$. So, x^2 will give a square whose side is $\sqrt{2}$. We shall have only one square for the area x^2.

(a) $x^2 = \sqrt{2} \cdot \sqrt{2}$



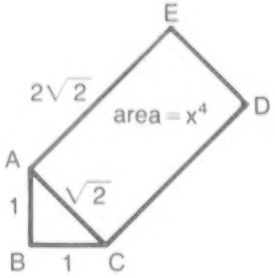
If, AB = BC = 1 then AC = $\sqrt{2}$ Completing the square with side AC we get ACDE as the square for x^2.

Step 2. For the area of x^4

x^4 can be written as $x \cdot x^3$; $x^2 \cdot x^2$ and $x^3 \cdot x$. Therefore,

(a) $x^4 = x \cdot x^3$
$(\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2})$
$= \sqrt{2} \cdot 2\sqrt{2}$

(a) $x^4 = \sqrt{2} \cdot \sqrt{2}$



AB = BC = 1
AC = $\sqrt{2}$
AE = $2\sqrt{2}$

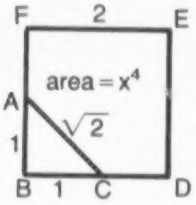
The area of the rectangle ACDE is $\sqrt{2} \cdot 2\sqrt{2}$ or area x^4

(b) $x^4 = x^2 \cdot x^2$

$= (\sqrt{2}\cdot\sqrt{2})\cdot(\sqrt{2}\cdot\sqrt{2})$

$= 2 \cdot 2$

(b) $x^4 = 2 \cdot 2$



$AB = BC = 1$

$AC = \sqrt{2}$

$BF = 2$

Completing the square with side BF, we get square BFED with area 2•2 or x^4.

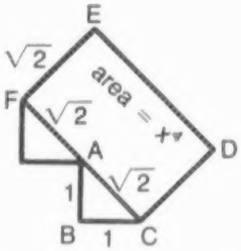
(c) $x^4 = x^3 \cdot x$

$= (\sqrt{2}\cdot\sqrt{2})\cdot(\sqrt{2}\cdot\sqrt{2})$

$= 2 \cdot 2 = 2\sqrt{x}\cdot\sqrt{2}$

Thus, x^4 will give 2 rectangles and 1 square. In total, x^4 will give 3 figures.

(c) $x^4 = 2\sqrt{2}\cdot\sqrt{2}$



$AB = BC = 1$

$AC = \sqrt{2}$

$CF = 2AC = 2\sqrt{2}$

$CD = AC = \sqrt{2}$

The area of the rectangle FCDE is $2\sqrt{2}\cdot\sqrt{2}$ or x^4

Step 3. For the area of x^6

x^6 can be written as $x \cdot x^5$; $x^2 \cdot x^4$; $x^3 \cdot x^3$; $x^4 \cdot x^2$ and $x^5 \cdot x$

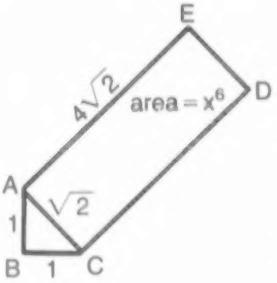
Therefore,

$x^6 = \sqrt{2}\cdot 4\sqrt{2}$

$= \sqrt{2}\cdot(\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2})$

$= \sqrt{2}\cdot 4\sqrt{2}$

(a) $x^6 = \sqrt{2}\cdot 4\sqrt{2}$



$AB = BC = 1$

$AC = \sqrt{2}$

$AE = 4AC = 4\sqrt{2}$

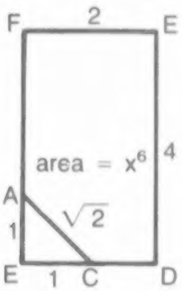
The rectangle ACDE has the area x^6 or $\sqrt{2}\cdot 4\sqrt{2}$

(b) $x^6 = x^2 \cdot x^4$

$= (\sqrt{2}\cdot\sqrt{2})\cdot(\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2})$

$= 2 \cdot 4$

(b) $x^6 = 2 \cdot 4$



$AB = BC = 1$

$AC = \sqrt{2}$

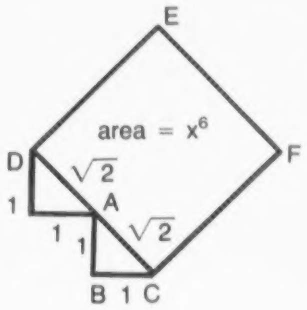
$BD = 2BC = 2$

$BF = 4BA = 4$

The rectangle BDEF has the area x^6 or 2 • 4.

(c) $x^6 = x^3 \cdot x^3$
$= (\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}) \cdot (\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2})$
$= 2\sqrt{2} \cdot 2\sqrt{2}$

(c) $x^6 = 2\sqrt{2} \cdot 2\sqrt{2}$

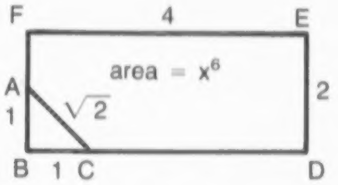


$AB = BC = 1$
$AC = \sqrt{2}$
$CD = 2AC = 2\sqrt{2}$

Completing the square with length DC, the square CDEF has the area x^6 or $2\sqrt{2} \cdot 2\sqrt{2}$.

(d) $x^6 = x^4 \cdot x^2$
$= (\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}) \cdot \sqrt{2} \cdot \sqrt{2})$
$= 4 \cdot 2$

(d) $x^6 = 4 \cdot 2$

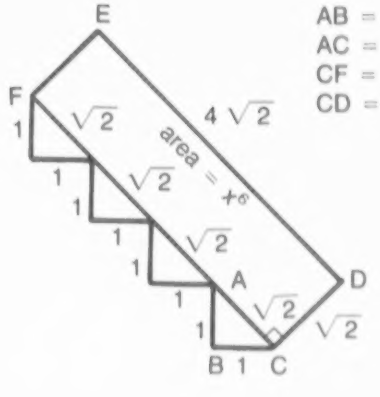


$AB = BC = 1$; $AC = \sqrt{2}$; $BD = 4BC = 4$;
$BF = 2AB = 2$
The rectangle BDEF has the area x^6 or $4 \cdot 2$.

(e) $x^6 = x^5 \cdot x$
$= (\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2}) \cdot \sqrt{2} \cdot \sqrt{2})$
$= 4\sqrt{2} \cdot \sqrt{2}$

x^6 will give 1 square and 4 rectangles. When totalled, x^6 will give 5 figures.

(e) $x^6 = 4\sqrt{2} \cdot \sqrt{2}$



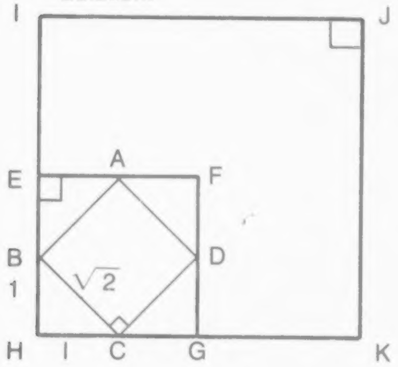
$AB = BC = 1$
$AC = \sqrt{2}$
$CF = 4CA = 4\sqrt{2}$
$CD = AC = \sqrt{2}$

Rectangle CDEF has area x^6 or $4\sqrt{2} \cdot \sqrt{2}$.

EXTENDED LEARNING:

1. Draw the geometric figures of the area x^8, when $x = \sqrt{2}$. How many figures do you have?
2. In one illustration draw the geometric figures of areas x^2, x^4, x^8, when $x = \sqrt{2}$.

Solution:



(1) The area of ABCD is x^2.
(2) The area of EFGH is x^4.
(3) The area of HIJK is x^8.

3. Draw the geometric figures of the areas x^3, x^5 and x^7 when $x = \sqrt{2}$. Do you get any squares? How many rectangles in each case do you get?

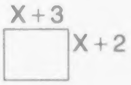
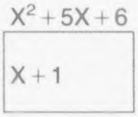
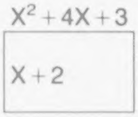
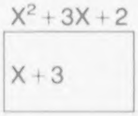
REFERENCE: Sloyer, Crouse E. *Mathematical Questions for the Classroom.* Boston: Pridle, Weber, Schmidt, Inc., 1977.

FACTORIZATION OF POLYNOMIALS RELATED TO AREA

GRADE LEVEL/ COURSE: Algebra I

INTRODUCTION/ OBJECTIVE: Given the area of a rectangle in the form of a polynomial, the students will be able to correlate the two factors of polynomial as the dimensions of the rectangle.

PROCEDURE: The students have already done factorization of polynomials of degree 2 and degree 3. They know, the area of a rectangle is the product of length and width. In the present lesson, the area of the rectangle is given in the form of a polynomial and the students will find the possible dimensions of the rectangle. If the area of a rectangle is given as a polynomial, in order to find the dimensions, the polynomial should be factored into two factors: the length and width. To introduce the lesson, students may begin this activity in class with the teacher's assistance and complete the work in class or at home if time is limited.

PROBLEMS		**SOLUTIONS**	
Area Given	**Factors**	**Possible Dimensions**	**Picture**
1. $X^2 + 5X + 6$	$(X + 2)\ (X + 3)$	$X + 2$ and $X + 3$	
2. $X^3 + 3X^2 + 14X + 6$	$(X + 1)\ (X + 2)\ (X + 3)$	(a) $X + 1$ and $(X + 2)\ (X + 3) = X^2 + 5X + 6$	
		(b) $X + 2$ and $(X + 1)\ (X + 3) = X^2 + 4X + 3$	
		(c) $X + 3$ and $(X + 2)\ (X + 1) = X^2 + 3X + 2$	

ACTIVITY: Students will be given some factorable polynomials and asked to give the dimensions of the rectangles.

Problem

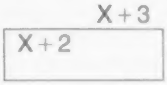
Two rectangles are equal in areas. The areas are $X^2 + 5X + 6$ and $X^2 + 6X + 5$. Find the numerical dimensions of each of the rectangles.

Solution

The areas are equal.
So, $X^2 + 5X + 6 = X^2 + 6X + 5$
Solving, we get $X = 1$

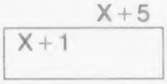
Dimensions for the 1st rectangle:

$X^2 + 5X + 6$ will give
a rectangle of dimensions
$(X + 3)(X + 2)$
Therefore, the dimension is 4 by 3.



Dimensions for the 2nd rectangle:

$X^2 + 6X + 5$ will give
a rectangle of dimensions
$(X + 5)(X + 1)$
Therefore, the dimension is 6 by 2.



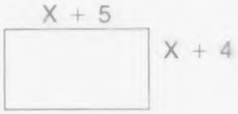
EXTENDED LEARNING:

Sample Problems

1. The area of a rectangle is $X^2 + 9X + 20$. Find the perimeter of the rectangle.

Solution:

The factors of $X^2 + 9X + 20$ are $X + 4$ and $X + 5$



$$\begin{aligned}\text{The perimeter} &= 2L + 2W\\ &= 2(X + 5) + 2(X + 4)\\ &= 2X + 10 + 2X + 8\\ &= 4X + 18\end{aligned}$$

2. Using the method in the case of area of triangle.

Example:

Given the area of a triangle in the form $X^2 - 7X - 30$ find the base and height of the triangle.

The area of triangle = ½ (base) (height), therefore,
2 (area) = base (height)

So, from the given problem, we get

$2(X^2 - 7X - 30) = \text{base (height)}$

$2(X - 10)(X + 3) = \text{base (height)}$

There are two possibilities:

$$(1)\quad \begin{cases} \text{base} = 2\,(X - 10) \\ \text{height} = X + 3 \end{cases}$$

$$(2)\quad \begin{cases} \text{base} = X - 10 \\ \text{height} = 2\,(X + 3) \end{cases}$$

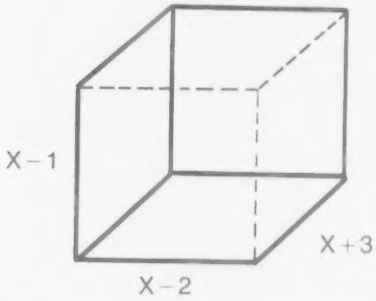
3. Using the polynomial of degree 3, as the volume of a solid, the dimensions, (length, width and height) can be determined.

Example:

The volume of a solid is $X^3 - 7X + 6$. Find the dimensions.

If we factor $X^3 - 7X + 6$, we get $X^3 - 7X + 6 = (X - 2)\,(X + 3)\,(X - 1)$

Volume = (length) (width) (height)



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FALLACIES IN PROOFS

GRADE LEVEL/ COURSE: Algebra II

INTRODUCTION/ OBJECTIVE: The purpose of this activity is to motivate students to appreciate mathematical proofs. The student will use critical analysis to determine whether or not the proof is valid. The advanced algebra student would not necessarily be given the specific reasons for each step of the proof.

PROCEDURE: The student is asked to read through the steps of the following proofs in order to determine the specific fallacies which obviously exist. The initial reading can be done individually followed by classroom instruction or it can be a small group project. Explanations of the fallacies are found after Activity 5.

ACTIVITY 1: **Theorem: $a = 1 \Rightarrow 2 = 1$ where $a \in R$.**

	Proof
1. $a = 1$	Given
2. $a(a-1) = 1(a-1)$	Multiplicative property of equality
3. $a \cdot a - a \cdot 1 = 1 \cdot a - 1 \cdot 1$	Distributive property
4. $a^2 - a = a - 1$	Definition of a power and multiplicative identity
5. $a^2 = 2a - 1$	Additive property of equality and distributive property
6. $a^2 = 2a - a$	Substitution
7. $a^2 = a$	Distributive property
8. $a^2 - 1 = a - 1$	Additive property of equality
9. $(a+1)(a-1) = a - 1$	Distributive property
10. $(a+1)(a-1) \cdot \frac{1}{a-1} = (a-) \cdot \frac{1}{a-1}$	Multiplicative property of equality
11. $a + 1 = 1$	Multiplicative inverse property
12. $1 + 1 = 1$	Substitution
13. $2 = 1$	Substitution

If the student cannot determine the fallacy in step #10 on his own, the teacher should offer suggestions or hints illustrating that the step involves division by zero.

Justification:

1. Division by zero is undefined because all well defined operations can be checked with the inverse operation which must be well defined:
   i.e., $a \div b = c$ implies $c \cdot b = a$.

   If $b = o$, then $a \div b = a \div o =$ undefined $= c$ implies $c \cdot o = a$.
   If $a \neq o$, then $c \cdot o \neq a$ because $c \cdot o =$ for all $c \in R$. Hence, there exists no real number c which could equal the quotient of a and b.
   If $a = o$, $c \cdot o = a = o$. But, for all $c \in R$, $c \cdot o = o$. Hence, the quotient c could be any real number, and in order for any operation to be well defined the answer must be unique.

2. A practical way to illustrate the impossibility of division by zero is to exhibit a sequence of quotients where each successive divisor is decreasing. 10 was used as a numerator simply as a matter of convenience. Any real number could have been substituted for the numerator and the divisor could have decreased in any manner. As the sequence is extended, the divisor becomes infinitesimally small, thereby producing a divergent sequence of quotients with no limit, i.e.:

$$\frac{10}{1,000,000} = .00001$$

$$\frac{10}{100,000} = .0001$$

$$\frac{10}{10,000} = .001$$

$$\frac{10}{1,000} = .01$$

$$\frac{10}{100} = .1$$

$$\frac{10}{10} = 1$$

$$\frac{10}{1} = 10$$

$$\frac{10}{\frac{1}{10}} = 100$$

$$\frac{10}{\frac{1}{100}} = 1,000$$

$$\frac{10}{\frac{1}{1,000}} = 10,000$$

$$\frac{10}{\frac{1}{10,000}} = 100,000$$

$$\frac{10}{\frac{1}{100,000}} = 1,000,000$$

etc.

After the student has studied the first theorem, determined the fallacy in the theorem, and reviewed the 'Justification', he or she may choose to analyze more of the following:

ACTIVITY 2: **Theorem: If** $\sqrt{-1} = \sqrt{-1}$**, then** $1 = -1$

(1) $\sqrt{-1} = \sqrt{-1}$

(2) $\sqrt{\frac{1}{-1}} = \sqrt{\frac{-1}{1}}$

(3) $\frac{\sqrt{1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{1}}$

(4) $\frac{1}{\sqrt{-1}} = \frac{\sqrt{-1}}{1}$

(5) $1 \cdot 1 = \sqrt{-1} \times \sqrt{-1}$

(6) Hence, $1 = -1$

Where did we go wrong?

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ACTIVITY 3: **Theorem: $b < a \Rightarrow a = b$ where $a, b, c \in R$, any real number is equal to any other real number where the second is less than the first.**

Proof

Let c be such that

1. $a = b + c$
2. $a(a-b) = (b+c)(a-b)$
3. $a^2 - ab = ab - b^2 + ac - bc$
4. $a^2 - ab - ac = ab - b^2 - bc$
5. $a(a-b-c) = b(a-b-c)$
6. $a(a-b-c) \cdot \left(\frac{1}{a-b-c}\right) = b(a-b-c) \cdot \left(\frac{1}{a-b-c}\right)$
7. $a \cdot 1 = b \cdot 1$
8. $\therefore a = b$

But remember, in the original hypothesis, b was given less than a.

Where did we go wrong?

ACTIVITY 4: **Theorem: Any positive real number is greater than itself.**

1. Given $a > o$ there exists some $b \in R$ where $b > o$ and $a > b$.
2. $ab > b^2$
3. $ab - a^2 > b^2 - a^2$
4. $a(b-a) > (b+a)(b-a)$
5. $a(b-a) \cdot \left(\frac{1}{b-a}\right) > (b+a)(b-a) \cdot \left(\frac{1}{b-a}\right)$
6. $a \cdot 1 > (b+a) \cdot 1$
7. $a > b + a$

From this theorem, since $b > o$, i.e., positive, a is not only greater than itself, but also greater than any number greater than itself.

Where did we go wrong?

ACTIVITY 5: **Theorem: All positive whole numbers are equal.**

By simply using the process of long division we have the following for all real numbers, x:

1. $\frac{x-1}{x-1} = 1$
2. $\frac{x^2-1}{x-1} = x+1$
3. $\frac{x^3-1}{x-1} = x^2+x+1$
4. $\frac{x^4-1}{x-1} = x^3+x^2+x+1$

Thus, $\frac{x^n-1}{x-1} = x^{n-1} + x^{n-3} + \ldots + x^2 + x + 1$

If we substitute 1 for x in all of the above identities, we find the right members to equal 1, 2, 3, . . ., n, respectively. The left member remains the same throughout.

Hence, $1 = 2 = 3 = 4 = 5 = \ldots = n$.

Where did we go wrong?

The Fallacies explained

The student should not be shown this section until he/she has *tried* to explain these by themselves.

Activity 2

Since the multiplicative inverse of a complex number is equal.

$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$ then $\frac{1}{\sqrt{-1}}$ in the left member $= \frac{1}{i} = \frac{0-i}{0^2+1^2} = -i$.

But the original expression in the left member is $\sqrt{-1}$ which equals i.

Activity 3

$a = b + c \Rightarrow a - b = c$

Hence, $a-b-c = o$, and in step #6 we are not following the rule, "Thou shalt not divide by zero".

Activity 4

Since $a > b \Rightarrow o > b - a$, we can conclude that $b - a < o$, i.e. a negative number. And in step #5 we are multiplying by the reciprocal of a negative number which is also negative. Hence the order of inequality should have been reversed.

Activity 5

For each of the identities in the theorem, $\frac{x^n-1}{x-1}$ equals $\frac{0}{0}$ whenever 1 is substituted for x. Again we see that $\frac{0}{0}$ can be any real number according to justification #1.

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FAREY SEQUENCE

GRADE LEVEL/ COURSE: Advanced Math

INTRODUCTION/ OBJECTIVE: The student will develop the Farey sequence and discover the following properties associated with the sequence.

Property 1: Any two consecutive terms a/b and c/d of any Farey sequence are related by $bc - ad = 1$.

Property 2: Fractions immediately before $^1/_2$ and its immediate successor of the Farey sequence are complementary.

Property 3: For any three consecutive terms a/b, c/d, e/f, then $\frac{c}{d} = \frac{a + 3}{b + f}$.

HISTORY: Farey was not a professional mathematician but a lawyer whose only claim to scientific fame is that in 1816 he discovered these sequences while perusing lengthy tables of decimal quotients.

ACTIVITY: **Step 1.** Develop the sequence.
The Farey sequence Fn of order n is defined as the ordered set consisting of 0/1, the irreducible proper fractions, with denominators from 2 to n, arranged in order of increasing magnitude, and 1/1.

F_1 $^0/_1, {}^1/_1$ (trivial)

F_2 $^0/_1, {}^1/_2, {}^1/_1$

F_3 $^0/_1, {}^1/_3, {}^1/_2, {}^2/_3, {}^1/_1$

F_4 $^0/_1, {}^1/_4, {}^1/_3, {}^1/_2, {}^2/_3, {}^3/_4, {}^1/_1$

F_5 $^0/_1, {}^1/_5, {}^1/_4, {}^1/_3, {}^2/_5, {}^1/_2, {}^3/_5, {}^2/_3, {}^3/_4, {}^4/_5, {}^1/_1$

F_6 $^0/_1, {}^1/_6, {}^1/_5, {}^1/_4, {}^1/_3, {}^2/_5, {}^1/_2, {}^3/_5, {}^2/_3, {}^3/_4, {}^4/_5, {}^5/_6, {}^1/_1$

Step 2. Note that any two consecutive terms, for example $^3/_4$ and $^4/_5$ in F_5 are related so that the product of the means is 1 more than the product of the extremes.

$4 \times 4 - 3 \times 5 = 1$. (Do more examples)

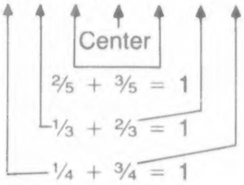
Generalize:

If a/b and b/c are two consecutive terms of any Farey sequence then $bc - ad = 1$.

Step 3. Note that fractions equidistant from $1/2$ are complementary, their sum equals one.

For example:

$F_6 = 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1.$



$2/5 + 3/5 = 1$

$1/3 + 2/3 = 1$

$1/4 + 3/4 = 1$

Step 4. Note that any three consecutive terms, for example $1/3$, $2/5$, $1/2$ in F_5 are related so that the sum of the numerators of the end fractions $1/3$, $1/2$ divided by the sum of their denominators produce the middle fraction $2/5$.

$\frac{1+1}{3+2} = \frac{2}{5}$ (Do more examples)

*This process is sometimes referred to as freshman addition.

Generalize:

For three consecutive terms of a Farey sequence a/b, c/d, e/f, then by Property 1:

i) $bc - ad = 1$
$de - cf = 1$

ii) $bc - ad = de - cf$ set equations equal

iii) $\frac{bc}{d} - \frac{ad}{d} = \frac{de}{d} - \frac{cf}{d}$ divide by d

iv) $\frac{bc}{d} + \frac{cf}{d} = e + a$ collect and reduce terms

v) $\frac{c\,(b+f)}{d} = e + a$ collect like terms

vi) $\frac{c}{d} = \frac{e+a}{b+f}$ divide by $(b + f)$

EXTENDED LEARNING:

1. Find missing term between $1/2$ and $3/5$ in a certain Farey sequence. There is exactly one term between. Find missing term, reconstruct whole sequence. Is more than one sequence possible? How many?

2. Try to establish relationship between π and number of lines in Farey sequence. (*See Teaching Secondary School Mathematics—page 346.)

REFERENCES:

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"FITTING" A CIRCLE INTO A PARALLELOGRAM *OR* DERIVING THE FORMULA FOR THE AREA OF A CIRCLE

GRADE LEVEL/ COURSE: 7

MATERIALS: Paper circles — 6" diameter, divided into 24 equal parts.

INTRODUCTION/ OBJECTIVE: Students will develop the concept of the area of a circle as the limit of the area of an inscribed polygon. While students may memorize the required formula for the area of a circle, and plug in the values, they often never comprehend the problem after securing the correct answer. The area of a circle is not as easily visualized as are areas bounded by squares, rectangles, triangles or trapezoids. By the time a student reaches the study of circular regions, he may even be tired of deriving formulas. Here is an activity even sophisticated thirteen-year-olds seem to enjoy that involves cutting and pasting.

PROCEDURES:

1. The study of the area of a circle logically follows perimeter of a circle and area and perimeter of rectangles, squares, parallelograms, trapezoids, and triangles. After careful development of all of the formulas, when the time comes for the study of circles, the class may be asked whether the area of a circular region can be adapted to the area of any previously studied.
2. Review the following information about circles with your students.
   1. A circle is a set of points equidistant from a given point called the center of the circle.
   2. The radius is the distance from the center of the circle.
   3. A diameter is a line segment that passes through the center and whose endpoints are on the circle. A diameter is also made of two radii. Therefore, $\underline{d = 2r}$.
   4. The ratio of the circumference to the diameter is identified as pi. Therefore,

      $$\frac{C}{d} = \pi \qquad \pi \cong 3.14 \text{ or } 3\tfrac{1}{7}$$

   5. The "distance around", the "perimeter of", or the "circumference of" the circle can be found from the formula $C = \pi d$ or $C = 2\pi r$.
3. The class should now be ready to discuss how a circle's area might be figured. When other areas of quadrilaterals and triangles were figured, students counted square units, rearranged pieces, and sometimes divided by two. Was there any way in which they could see how a circle could be fitted into an already studied polygon?
4. Since a circle is not composed of straight line segments, it is not possible to define area so simply. However, if a circle were divided into equal parts and chords drawn as in figure 1, it would be possible to define the area as the sum of the areas of the triangles into which the polygon had been partitioned.

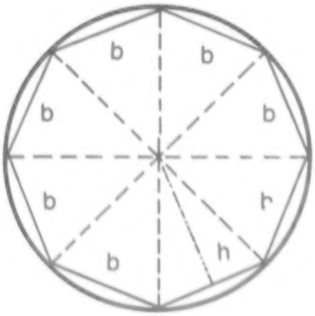


Figure 1

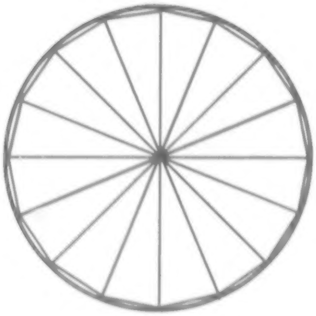


Figure 2

The smaller the triangles that are inscribed within the circle, the more nearly they would fill it up. As these triangles shrink, their height would get closer to the length of the radius and their bases would nearly sit on the circumference. (Figure 2)

Then the area could be found by multiplying the number of triangles, n, by the area of a triangle or $A = n\frac{1}{2}bh$. The height is the radius and the number of bases is the circumference.

Therefore, $A = n\frac{1}{2}bh$
$A = \frac{1}{2}Ch$ $\quad (nb = C)$
$A = \frac{1}{2}Cr$ $\quad (h = r)$
$A = \frac{1}{2}(2\pi r)r$ $\quad (C = 2\pi r)$
$A = \pi r^2$

Can these "triangles" be further arranged?

ACTIVITY: **Step 1.** Give each student a 6" circle with the center clearly defined and divided into twenty-four equal parts. (This is a reasonable number to work with) See Figure 3 (not drawn to scale).

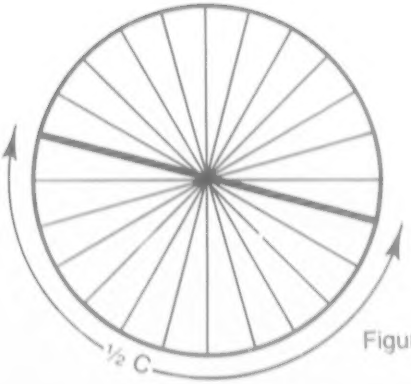


Figure 3

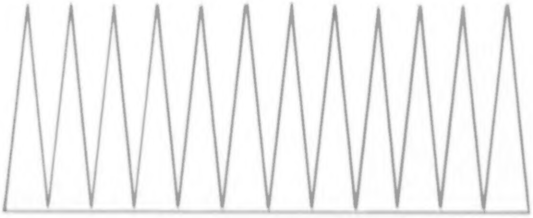
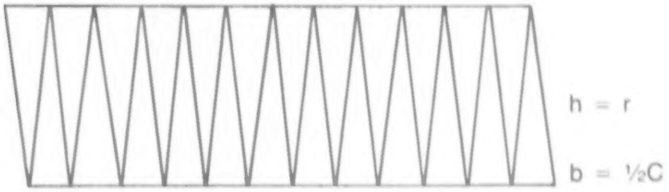


Figure 4

Step 2. Cut along a diameter. Beginning at the center each time, cut along the radii, coming very close to the edge of the circular regions but not cutting the edge. This results in a string of pie-shaped wedges as shown in Figure 4.

Figure 5



$C = \pi d$
$C = 2\pi r$

Step 3. Paste the string of pie-shaped wedges onto a different piece of colored paper in such a way as to suggest a parallelogram as in Figure 5. Note that the length of this region is ½ the circumference of the circle or $\frac{1}{2}C$ and the height is the radius of the circle or r.

Step 4. Paste the second string of wedges on the first as shown in figure 5 so that the region is covered and the wedges are not overlapping. Note that the circular region has now been adapted to a parallelogram and it is possible to relate the area of the parallelogram to the circle. $A = bh$. Since $b = \frac{1}{2}C$ and $h = r$, the new formula is $A = \frac{1}{2}Cr$.

Remember that $C = 2\pi r$, so . . .

$$A = \frac{1}{2}Cr$$
$$A = \frac{1}{2}(2\pi r)r$$
$$A = \pi r \cdot r$$
$$A = \pi r^2$$

Therefore, we have "fitted" the area of a circular region within the area of a parallelogram.

EXTENDED LEARNING:

1. Have the students comment on this derivation activity. Explain that the Greeks labored long on this concept and spent much time in trying to prove it logically.
2. Have the students figure the area of a circle using $A = \pi r^2$. Then using the same circle drawn to scale, reassemble it into a parallelogram and use the formula $A = bh$. Compare areas.
3. Have the students try assembling circles into parallelograms dividing the circles into different equal parts such as 36, and figuring the areas both ways.

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FOLDING PAPER CONIC SECTIONS

GRADE LEVEL/ COURSE: Algebra II, Geometry

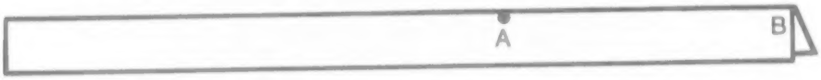
MATERIALS: Paper, Pencils, Scissors

INTRODUCTION/ OBJECTIVE: Students will develop a better understanding of conic sections by using paper folding techniques.

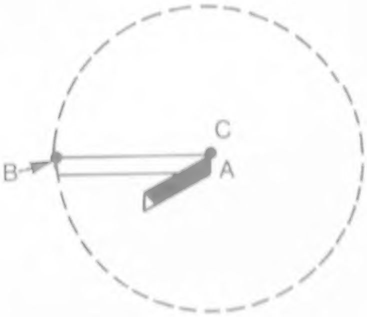
PROCEDURE:

1. The teacher should construct the conic sections before presenting the activities to the students.
2. The students may work individually or with a partner.
3. The teacher should guide the class through each step of the activity.

ACTIVITY 1: Instruct the class to mark a point C at or near the center of a piece of plain paper (approximately 8½" × 11"). Then make a 1 inch fold along the bottom (narrow) edge of the paper, crease and tear off. Then fold the 1 inch strip in half lengthwise. Now mark point A on the creased edge of this strip 2 inches to 4 inches from one end and mark point B at this end of the strip, also on the creased edge. See diagram. Now place point A at point C and fold the larger paper so that the fold line runs along the short end of the strip at point B. Mark this fold line with a pencil. Continue to move the strip around keeping point A at point C and always folding the larger paper along the line of the strip at point B. Repeat until at least 20 creases and lines have been marked. A circle will gradually appear as the creases are marked.

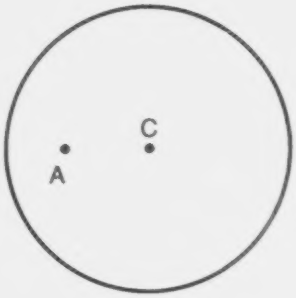


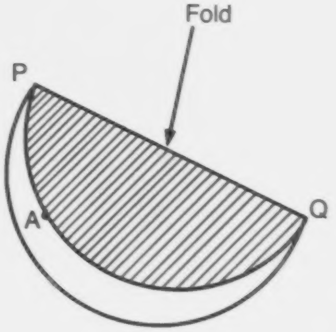
Point out that the line segment $\overline{AB}$ is the measure of the radius of the circle with center C. The crease lines are tangents to the circle.

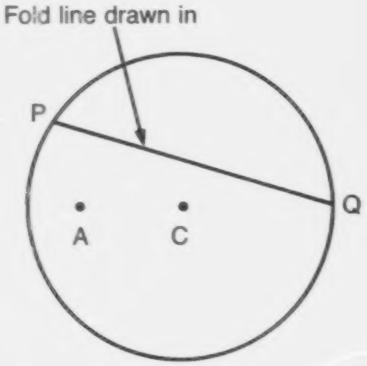


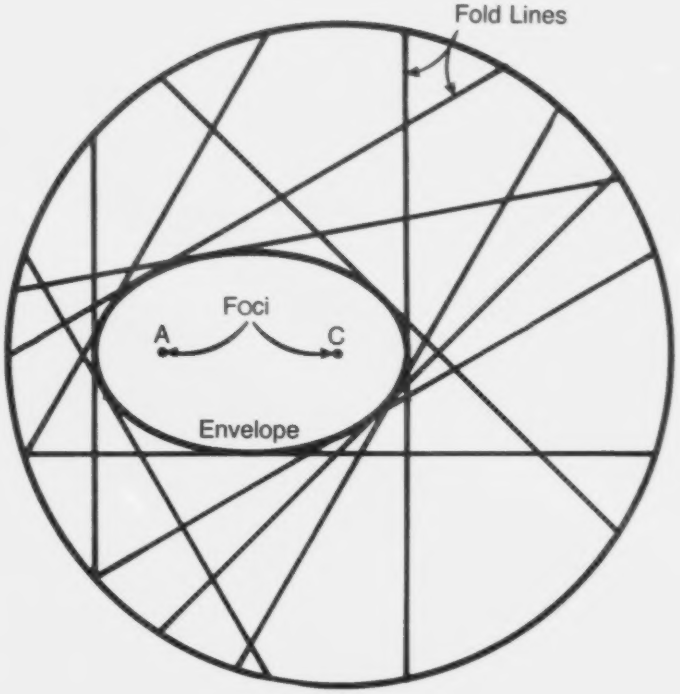
ACTIVITY 2: Instruct the class to take a piece of plain paper and draw a large circle with a radius of approximately 4″. Cut out the circle carefully and on it mark a point A about 1″ from the edge. Point out that precise lengths are not critical but these suggested lengths give good results. Now fold the paper circle along any line PQ so that the edge of the circle just touches the point A. Draw in the fold line after unfolding the circle. Keep repeating this process of folding and marking the fold lines. An ellipse will gradually appear surrounded by all the fold lines.

The lines are tangents to the ellipse and form an "envelope" for it.









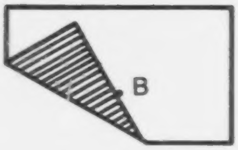
Students may now investigate the result of taking A nearer the center of the circle C. Ask students to see what happens if A and the center C are the same point.

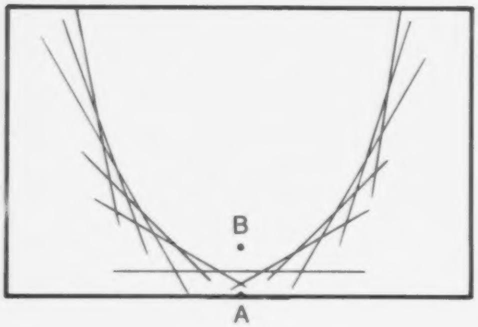
Point out that A and the center C are symmetrically placed and are called the foci of the ellipse.

ACTIVITY 3: Instruct the class to take a piece of plain paper (approximately 8½" × 11") and fold to find the midpoint A of one of the longer edges. Mark a point B that is about one inch above the midpoint of this edge. Fold the paper so this edge touches point B. Mark a sharp crease. Open the paper and use the pencil to "draw" the fold line.

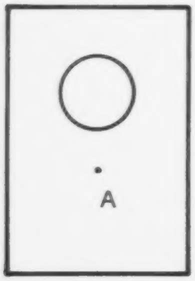
Fold again in a slightly different direction making sure that the lower edge touches the point B. Repeat this about twenty times, always marking the fold. A parabola will appear.

Point out to the class that the point B is the focus of the parabola, line AB is the axis of symmetry. The crease or fold lines are tangents to the parabola.

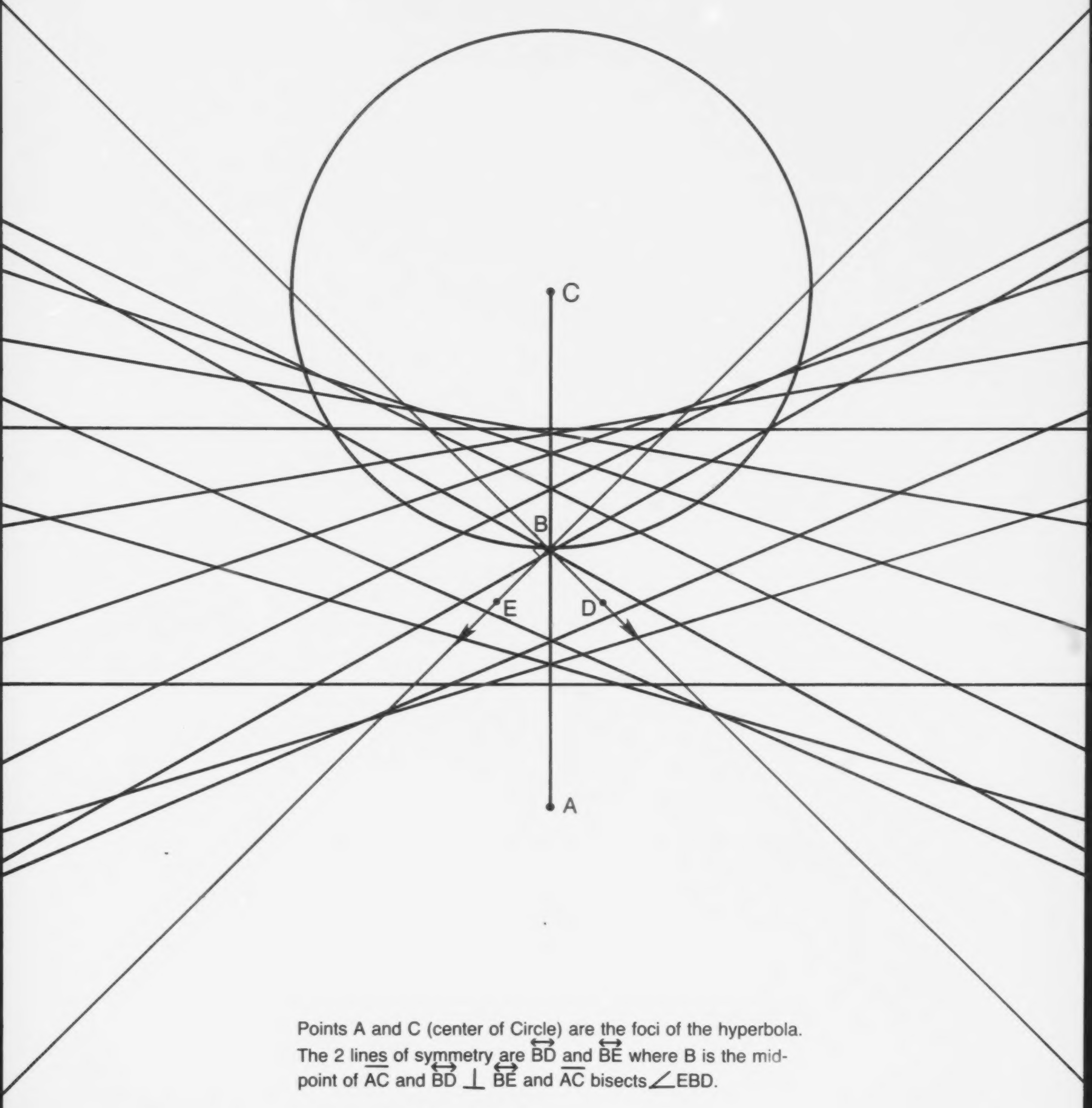




ACTIVITY 4: Instruct the class to take a piece of plain paper — preferably transparent or tracing paper and draw a circle with a radius of about 4". Mark a point about 2" below the circle. Now fold the paper so that this point (which we will label A) falls on the circle and make a sharp crease. Open the paper and use a pencil to mark the crease. Fold in a different direction so that the point A again falls on the circle. Mark each crease. Continue this process until the point has been folded into different positions all the way around the circle.



Hyperbola



Points A and C (center of Circle) are the foci of the hyperbola. The 2 lines of symmetry are $\overleftrightarrow{BD}$ and $\overleftrightarrow{BE}$ where B is the midpoint of $\overline{AC}$ and $\overleftrightarrow{BD} \perp \overleftrightarrow{BE}$ and $\overline{AC}$ bisects $\angle EBD$.

Show the students where the two foci of the hyperbola are located. Indicate that the hyperbola has two lines of symmetry.

EXTENDED LEARNING: Give the students sheets of plain paper, pencils, scissors, and tracing paper and ask them to construct the following using the methods described in the unit:

a) circle, b) ellipse, c) parabola, d) hyperbola.

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Bolt, Brian. *Mathematical Activities.* New York: Cambridge University Press, 1982.

FOUR FOURS

GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVE: The student will use four 4s to represent natural numbers.

PROCEDURE: The teacher may use this activity as an independent activity after verifying the students' understanding of all operations listed under 1 b). The teacher may use examples listed beside zero to ensure understanding of the rules.

ACTIVITY:

Step 1. Write representations for the numbers 0 to 10 using four 4s and the following rules:

a) Use all four 4s in each representation of a natural number.

b) You may use any combination of the following operations:

1) $+$, $-$, $\times$, $\div$

2) $\sqrt{}$

3) exponents of 4: 4^4 or $4^{\sqrt{4}}$

4) decimals: .4, 4.4

5) symbols of grouping: ( ), [ ]

Step 2. For any given natural number, use more than one representation using four 4s.

Examples:

$0 = (4 - 4) + (4 - 4);\ (4 + 4) - (4 + 4);\ 4^4 - 4^4$

$1 = \frac{4 + 4}{4 + 4};\ \frac{4}{4} \div \frac{4}{4};\ \frac{4 \times 4}{4 \times 4};\ \frac{4 \div 4}{4 \div 4}$

$2 = 4 - \frac{4 + 4}{4};\ \frac{4 \times 4}{4 + 4};\ \frac{4}{\sqrt{4}} + (4 - 4)$

$3 = \frac{4 + 4 + 4}{4};\ (4 - \sqrt{4}) + \frac{4}{4}$

$4 = \frac{4 - 4}{4} + 4;\ \frac{4}{4} \times (\sqrt{4} + \sqrt{4})$

$5 = \frac{4 \times 4 + 4}{4};\ \sqrt{4} + \frac{4}{4} + \sqrt{4};\ \frac{4}{4} + \sqrt{4 \times 4}$

$6 = 4 + \frac{4 + 4}{4};\ \frac{4 \times 4 - 4}{\sqrt{4}};\ 4! \div 4 \times \frac{4}{4}$

$7 = 4 + 4 - \frac{4}{4};\ 4! \div 4 + \frac{4}{4}$

$8 = 4 + 4 + 4 - 4;\ \frac{4 \times \sqrt{4}}{4} \times 4$

$9 = \frac{4}{4} + 4 + 4;\ \frac{4 \times 4 + \sqrt{4}}{\sqrt{4}}$

$10 = \frac{4! - \sqrt{4} - \sqrt{4}}{\sqrt{4}};\ 4 + 4 + \frac{4}{\sqrt{4}};\ \sqrt{4} + \sqrt{4} + \sqrt{4} + 4$

EXTENDED LEARNING:

1. Represent each of the natural numbers from 11 to 30 using four 4s.

2. Use the rules stated in the procedure and represent numbers 0 to 10 using the digits 1972, or four 9s.

GRAPHING AND SPATIAL RELATIONS

GRADE LEVEL/ COURSE: 7–9

MATERIALS: Pencil, paper, initial graph design, a variety of coordinate planes, a straight edge, and transparencies of graph design and coordinate planes for teacher to use on the overhead projector.

INTRODUCTION/ OBJECTIVE: The student will predict and graph the effect that changes in values of x and y will have on a given shape on a coordinate plane.

PROCEDURE: This activity is for students who have mastered graphing of ordered pairs. The teacher should work with the entire class to present the initial design and give directions for the design transfer. Students may do the design transfers in class or at home. Class discussion should follow on how the area of the design has changed on each graph.

ACTIVITY:

Step 1. Give the students an initial design and a variety of coordinate planes, (see accompanying example).

Step 2. Instruct the students to label the points on the initial design as ordered pairs.

Step 3. Have students plot these same points on the variety of provided coordinate planes.

Step 4. Ask the students the following questions. Students will make a prediction and then graph the figure to support their answers.

a) When the spacing between points is doubled on both the x and y axis, what effect will this have on the area of the given shape? A simpler design such as a rectangle could be plotted to show that the number of square units remains the same but if both the length and width are doubled in measurement then the measurement of the area is multiplied by four. Students could also plot triangles and measure to see that this relationship still applies.

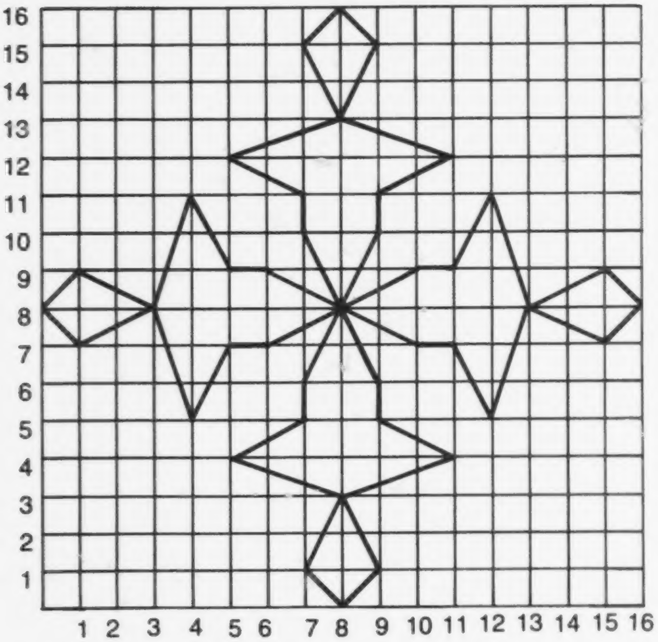
b) When the spacing between the points are only doubled horizontally how will the area be affected? Again beginning with a rectangle and/or triangular shape and changing the horizontal dimension only, the students may better be able to see that as one dimension is doubled the measured area is doubled.

c) When the spacing between the points are only doubled vertically how will the area be affected? As in question #2, as one dimension is doubled the measured area will be doubled.

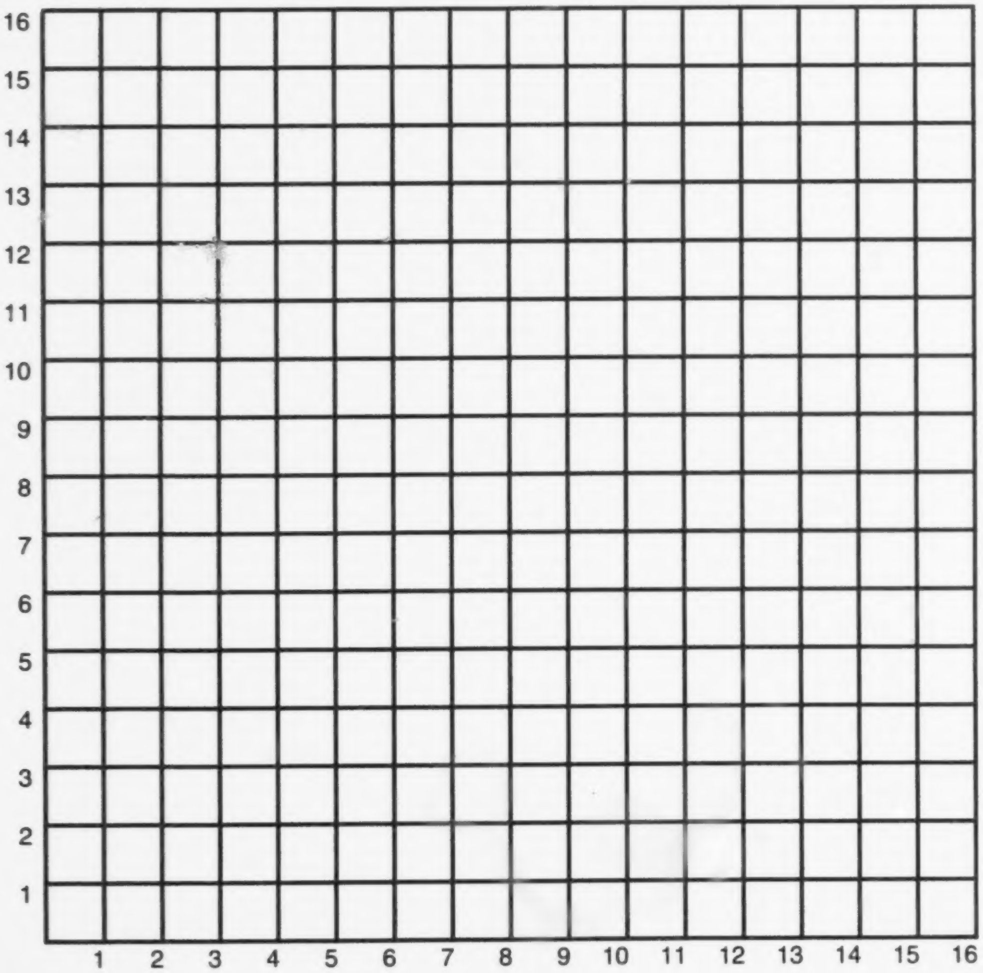
d) How will a change in the angle of the axes affect the area of the design? Discuss the difference in the area of the square unit which was used in the original plane and the area of a rhombus which is created by slanting the x and y axes. Since the area of a rhombus is equal to the base times the height, a decrease in the height of the rhombus would bring a proportional decrease in the area of the rhombus.

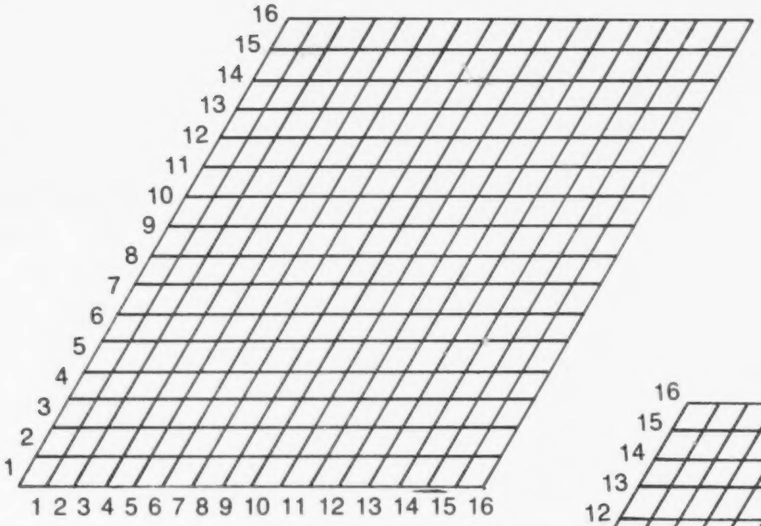
EXTENDED LEARNING:

1. Have students create their own designs listing the directions as a set of ordered pairs. Have students exchange directions and graph one anothers designs on various planes.
2. Have students create a variety of coordinate planes different from those given.
3. Have students with computer programming knowledge enter designs into the computer for display.



Transfer the snowflake to each new graph keeping the end points for each line segment the same in every transfer. Observe how the snowflake changes.

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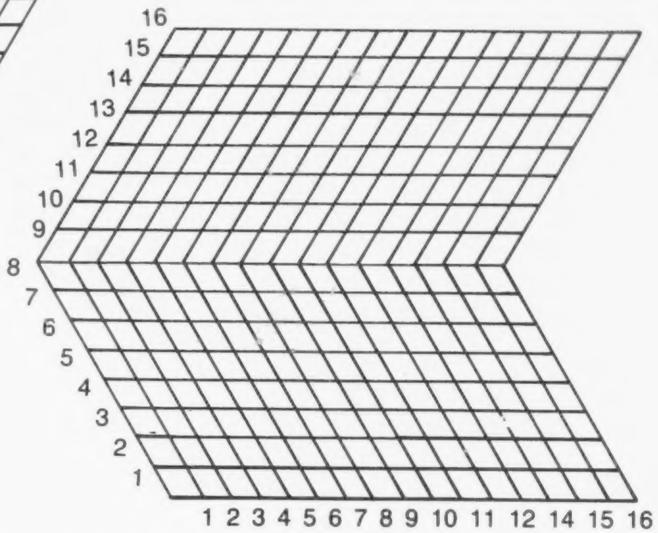
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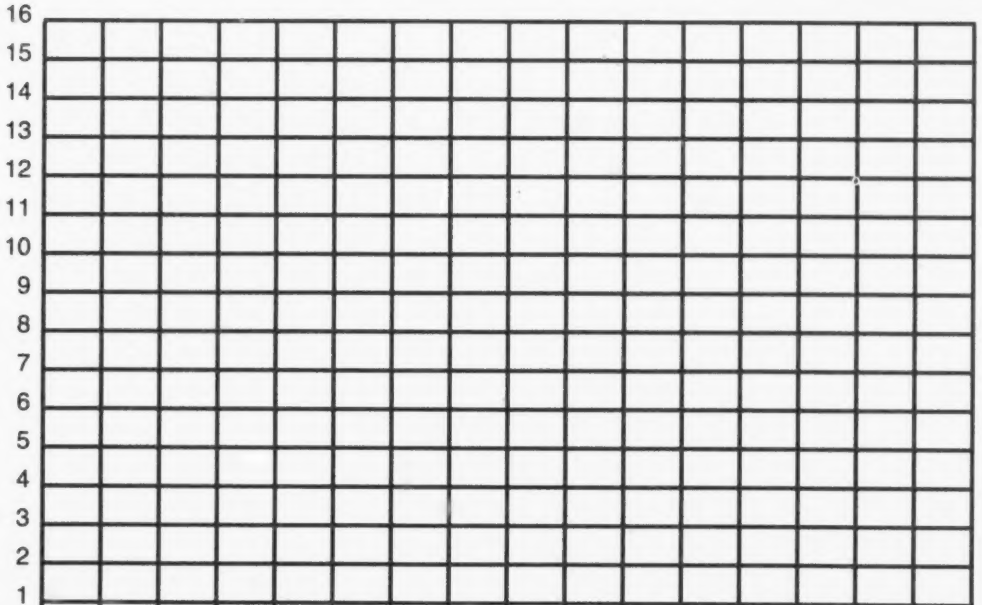
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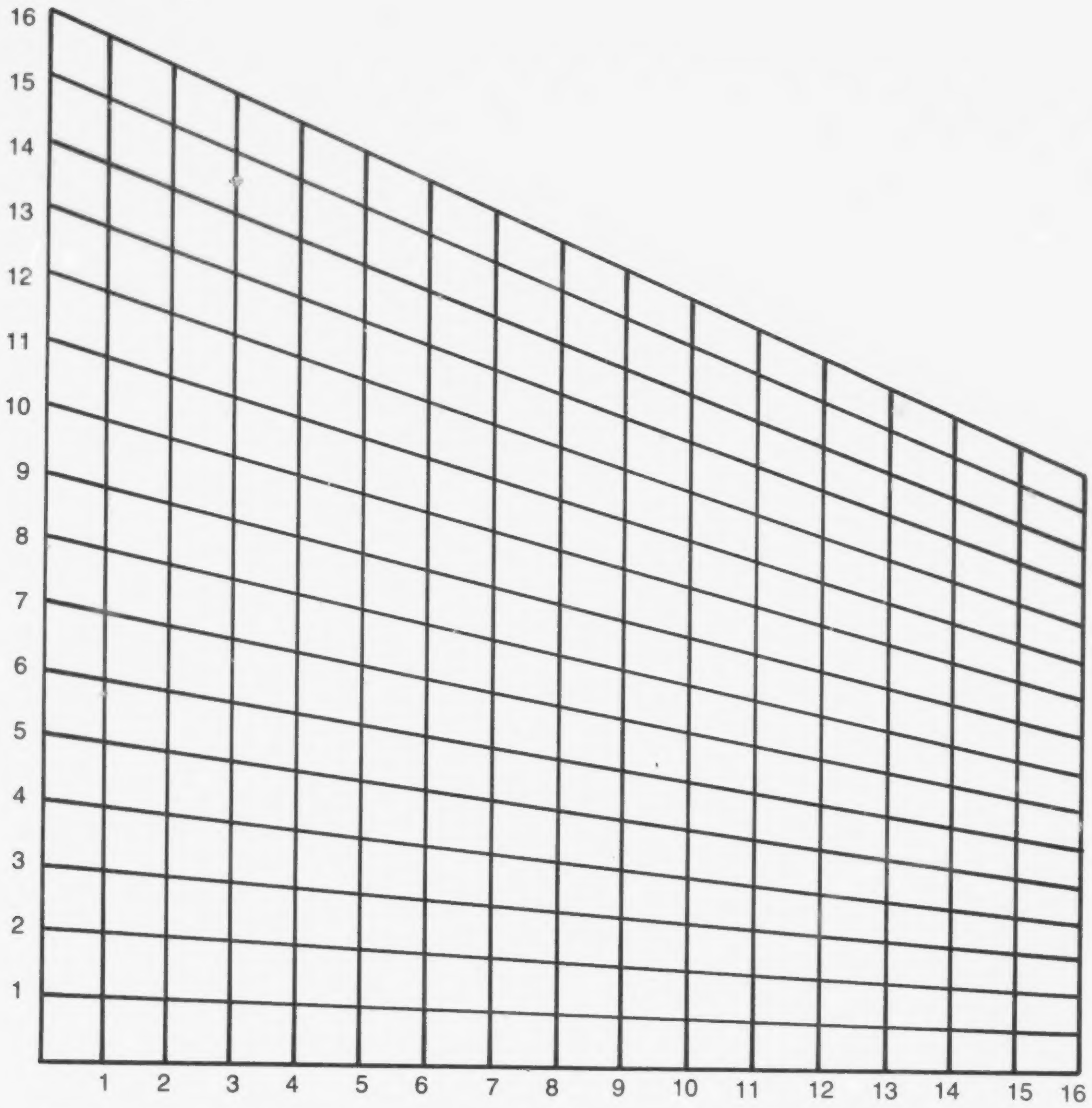
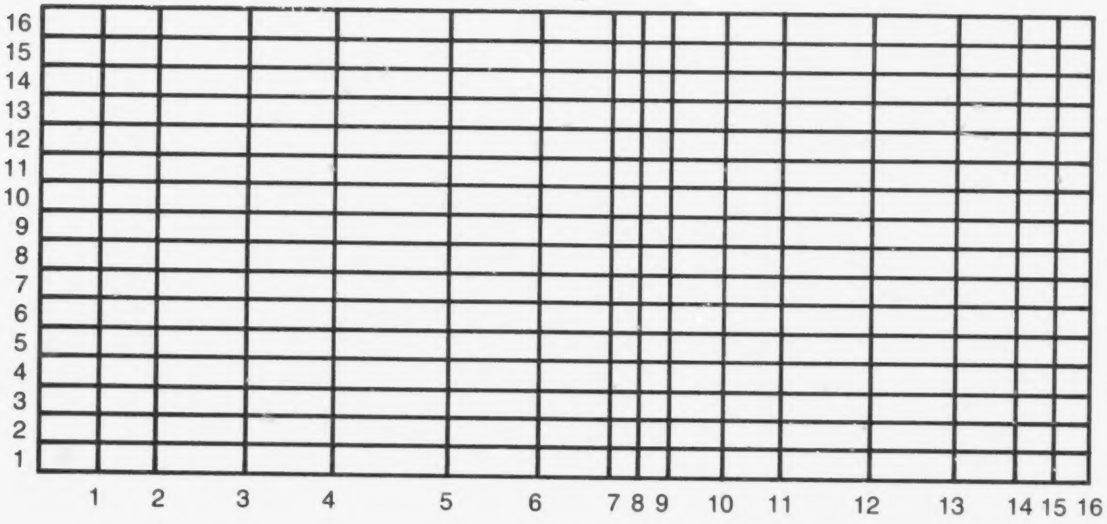




GREATEST COMMON FACTOR OF MORE THAN TWO NUMBERS

GRADE LEVEL/ COURSE: 7–8, Algebra I

MATERIALS: Paper, pencil, and worksheets.

INTRODUCTION/ OBJECTIVE: Sometimes students have difficulty finding the prime factorization of large numbers while solving GCF problems. This method encourages the students to find GCF without prime factorization. The students will be able to find the GCF of any set of numbers without prime factorization.

PROCEDURE: Work with the class to present examples and allow students to work independently on assigned numbers.

ACTIVITY: The GCF will be found by a division method. The idea behind it is as follows:

If G_1 is the GCF of a and b, and if G_2 is the GCF of G_1 and c, then G_2 is the GCF of a, b, c.

The following example determines the GCF of 165, 198, and 275.

$$\begin{array}{rll} & 1 & \\ 165 & \overline{)198} & \\ & \underline{165} & 5 \\ & 33 & \overline{)165} \\ & & \underline{165} \\ & & 0 \end{array}$$

33 is the GCF of 165 and 198

$$\begin{array}{rll} & 8 & \\ 33 & \overline{)275} & \\ & \underline{264} & 3 \\ & 11 & \overline{)33} \\ & & \underline{33} \\ & & 0 \end{array}$$

11 is the GCF of 165, 198 and 275

Comment: Continue to divide the remainder into the original divisor until a zero remainder is obtained. The last nonzero remainder is the GCF.

Similar problems can be solved in the classroom. Assign the following sets of numbers; 34, 51, 85; 45, 108, 144; 65, 104, 143.

Find the GCF of $x^2 - 1$, $x^3 - 1$, $x^4 - 2x^2 + 1$

$$\begin{array}{rll} & x & \\ x^2-1 & \overline{)x^3-1} & \\ & \underline{x^3-x} & x+1 \\ & x-1 & \overline{)x^2-1} \\ & & \underline{x^2-x} \\ & & x-1 \\ & & \underline{x-1} \\ & & 0 \end{array}$$

So, $x-1$ is the GCF of x^2-1 and x^3-1.

$$
\begin{array}{r|l}
\multicolumn{1}{r}{} & x^3+x^2-x-1 \\ \cline{2-2}
x-1 & x^4-2x^2+1 \\
\multicolumn{1}{r}{} & x^4-x^3 \\ \cline{2-2}
\multicolumn{1}{r}{} & x^3-2x^2 \\
\multicolumn{1}{r}{} & x^3-x^2 \\ \cline{2-2}
\multicolumn{1}{r}{} & \quad -x^2+1 \\
\multicolumn{1}{r}{} & \quad -x^2+x \\ \cline{2-2}
\multicolumn{1}{r}{} & \qquad -x+1 \\
\multicolumn{1}{r}{} & \qquad -x+1 \\ \cline{2-2}
\end{array}
$$

So, $x-1$ is the GCF of x^2-1, x^3-1, and x^4-2x^2+1

REMARKS: Practicing this method, the students will be able to find the GCF of numbers that contain more than three digits.

GROUPS

GRADE LEVEL/ COURSE: Algebra II and above

INTRODUCTION/ OBJECTIVE: The student will use some of the field axioms with a system (i.e., group) other than the real numbers.

PROCEDURES:

1. Using some of the field axioms for real numbers, we will define a group and give two examples.
2. A *group* is a system consisting of elements $x, y, z, \ldots$ of a set S and a binary operation $*$ such that for all elements in the set the following properties are true.
   - **(a)** $x * y \in S$ [closure axiom]
   - **(b)** $(x * y) * z = x * (y * z)$ [associative axiom]
   - **(c)** there exists a unique element $e \in S$ such that for any $X \in S$
     $x * e = x$ [identity axiom]
   - **(d)** for each $x \in S$ there exists $x \in S$ such that $x * x = e$ [inverse axiom]

ACTIVITY 1: Show that the set $S = \{a, b, c\}$ with the operation defined by this table is a group.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

If the four axioms can be verified, then set S and operation $*$ constitute a group.
(a) By observing the table above we see that the closure axiom is satisfied.

(b)

$$\begin{array}{rclcrcl}
(a * b) * c &=& a * (b * c) & \quad & (a * c) * b &=& a * (c * b) \\
b * c &=& a * a & & c * b &=& a * a \\
a &=& a & & a &=& a \\
(b * a) * c &=& b * (a * c) & & (b * c) * a &=& b * (c * a) \\
b * c &=& b * c & & a * a &=& b * c \\
a &=& a & & a &=& a \\
(c * a) * b &=& c * (a * b) & & (c * b) * a &=& c * (b * a) \\
c * b &=& c * b & & a * a &=& c * b \\
a &=& a & & a &=& a
\end{array}$$

The six cases above show that the associative axiom is satisfied.

(c) The identity element is "a" since
$a * a = a$
$a * b = b$
$a * c = c$

(d) Each element of the set has an inverse: a is its own inverse while b and c are inverses of each other.

Thus, set S and operation $*$ constitute a group.

ACTIVITY 2: Show the set S = {1, − 1, i, − i} is a group with respect to multiplication.

A multiplication table for this set would like this.

X	1	− 1	i	− i
1	1	− 1	i	− i
− 1	− 1	1	− i	i
i	i	− i	− 1	1
− i	− i	i	1	− 1

(a) Again by observing the table we see the set is closed under this operation.

(b) Multiplication is associative (Why?)

(c) The identity element in the set is 1.

(d) Each element has an inverse: 1 is its own inverse, − 1 is its own inverse, i and − i are inverses of each other.

EXTENDED LEARNING:

1. Show that the set {0, 1, 2} with addition mod 3 is a group. Is it an Abelian group? (In an Abelian group the commutative property must also hold for all elements.
2. Find other examples of groups.

HAPPY NUMBERS

GRADE LEVEL/ COURSE: 4–6, 7, 8

INTRODUCTION/ OBJECTIVE: The students will determine numbers that are Happy Numbers.

PROCEDURE: This activity may be used when students are proficient at the basic computation skills of addition and multiplication. The activity will begin with teacher demonstration and conclude with student independent work by students.

ACTIVITY:

Step 1. The teacher should explain the process used in finding Happy Numbers using the following examples:

Start with 13, then square the digits in the number and find the sum of the squares:

$1^2 + 3^2 =$

$1 + 9 = 10$

Repeat the squaring process for 10

$1^2 + 0^2 =$

$1 + 0 = 1$

The starting natural numbers for which the above pattern ends in a 1 are called Happy Numbers. Thus 13 is a Happy Number.

For lower grade students who are not familiar with squaring numbers, the teacher will need to explain the process simply by saying, "To square a number means to multiply it by itself."

5^2 means to multiply 5×5, so $5^2 = 25$

Step 2. Follow with a second demonstration:

Is 4 a Happy Number?

$$
\begin{aligned}
4^2 &= 16 \\
1^2 + 6^2 &= 1 + 36 = 37 \\
3^2 + 7^2 &= 9 + 49 = 58 \\
5^2 + 8^2 &= 25 + 64 = 89 \\
8^2 + 9^2 &= 64 + 81 = 145 \\
1^2 + 4^2 + 5^2 &= 1 + 16 + 24 = 42 \\
4^2 + 2^2 &= 16 + 4 = 20 \\
2^2 + 0^2 &= 4 + 0 = 4
\end{aligned}
$$

And the sequence repeats.

4 is not a Happy Number because the procedure never produces a number containing 1 in the units place.

Step 3. Have the students work independently to determine which of the following are Happy Numbers:

7	70	10	58
11	83	36	32

Note: 56, 11, 83 and 36 are not Happy Numbers.

Which years were Happy Years?

1900 1969 1985

EXTENDED LEARNING: Can you determine other Happy Numbers?

REFERENCES: Bezusha, Stanley. Contemporary Motivated Mathematics Book 1. Chestnut Hill, Mass.: Boston College Press. 1972.

HEXAGONAL FLIP

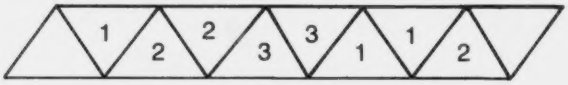
GRADE LEVEL/ COURSE: 7–8

MATERIALS: Worksheet, pencil, scissors, colored markers, and glue.

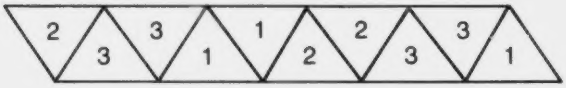
INTRODUCTION/ OBJECTIVE: The student will extend his understanding of two dimensional geometric shapes; he will explore the relationships among paralellograms, regular hexagons, and equilateral triangles. He may be encouraged to investigate, through analytical reasoning, the areas of regular polygons.

PROCEDURE: Distribute the materials and have students work in class and individually as the teacher guides them through this activity. The teacher should make a sample hexagonal flip prior to the instruction so that the student can see the finished product and the teacher can anticipate the student's difficulties.

ACTIVITY: **Step 1.** Number the triangles on the front side of the paper as follows:



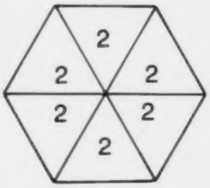
Step 2. Note the * in the *lower right* corner of your work sheet. Flip the work sheet so that the * is now on the back side in the *upper right* corner and number the triangles facing up as follows:



Step 3. On both sides of the paper, use a marker to color all triangles which are numbered 1; use a contrasting marker to color all triangles which are numbered 2; use another contrasting marker to color all triangles which are numbered 3.

Step 4. Cut out the parallelogram.

Step 5. Fold on the lines to form a hexagon, being careful to have the same color in each triangle which faces up, the same color in each triangle which faces down, and the same color in each hidden triangle. (See sketch)



Step 6. Overlap the two unmarked triangles so that the unmarked areas face each other and glue those two triangles together.

Step 7. Once the glue is dry, the hexagon can be turned inside-out repeatedly, giving a different color with each turn.

EXTENDED LEARNING:

At the pre-geometry level the student will be able to visualize certain relationships between the hexagon and equiangular triangles. He will note that a regular hexagon consists of six congruent equiangular triangles and that these six equiangular triangles which radiate from the center of the hexagon join with no overlap or gaps.

The geometry student will—

(a) correlate the area of regular hexagons and equiangular triangles;

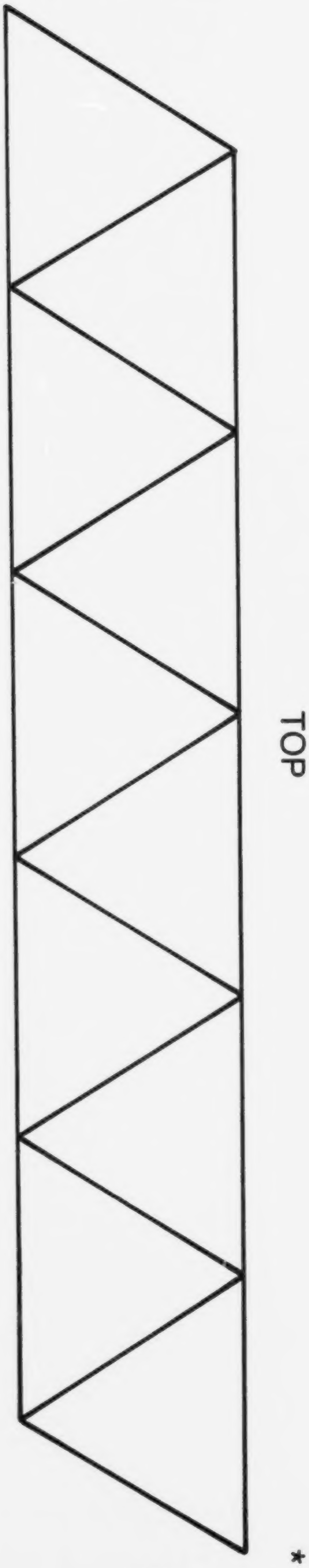
(b) note that the sum of the six central angles equals 360°;

(c) name a segment from the center perpendicular to the base of the hexagon to be the apothem;

(d) observe that the apothem will create two 30°-60°-90° triangles whose properties will be reviewed and applied;

(e) note that the sides of the triangles which radiate form radii of the hexagon.

WORK SHEET



INVESTIGATING SQUARES AND CUBES

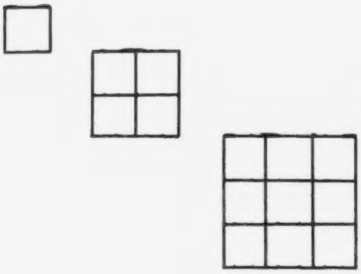
GRADE LEVEL/ COURSE: 7–8

INTRODUCTION/ OBJECTIVE: Students will discover some interesting number and geometric patterns by investigating certain relationships among perfect squares and cubes. Students should be familiar with squares and cubes of integers, with the use of exponents, and with drawing squares.

PROCEDURE: Interaction between teacher and entire class.

ACTIVITY 1: Have students construct a table of the squares of the first ten positive integers, drawing pictures to represent each:

Number	Square
1	1
2	4
3	9
4	16
5	25
6	36
7	49
8	64
9	81
10	100



etc.

Have students look at each drawing and identify any shapes or figures (rectangles, squares) that they see inside them. For example, the drawing for $5^2 = 25$ contains three 2×2 squares, one 3×3 square, and 4 left over boxes. If we look at the *number* 25, however, we see that it is made up of the sum of 9 and 16, that, $3^2 + 4^2$. Have students look for additional sums of this sort. They will probably notice that $100 = 36 + 64$ or $6^2 + 8^2$.

Next, include a column in your table for cubes. Ask students to find a square number that is comprised of squares and cubes, or cubes only. Possible responses are: $36 = 9 + 27 = 3^2 + 3^3$ (if we continue the table, we find that $15^2 = 225 = 100 + 125 = 10^2 + 5^3$).

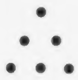
Other responses are: $36 = 1 + 8 + 27 = 1^3 + 2^3 + 3^3 = 6^2$ and

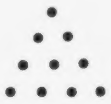
$$100 = 1 + 8 + 27 + 64 = 1^3 + 2^3 + 3^3 + 4^3 = 10^2$$

Direct students' attention to the bases in the previous expressions. Elicit the pattern and a few more expressions of this type. Arrange the expressions as follows and ask what geometric figure is suggested by this arrangement (triangle).

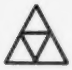
$$1 = 1$$
$$1 + 2 = 3$$
$$1 + 2 + 3 = 6$$
$$1 + 2 + 3 + 4 = 10$$
$$1 + 2 + 3 + 4 + 5 = 15$$

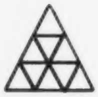
ACTIVITY 2: Looking at the sequence 1, 3, 6, 10, 15, 21 . . ., guide students into drawing a triangle made up of points for each of the numbers in this sequence.



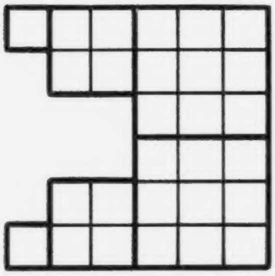


Students will probably agree that *Triangular Numbers* is a good name for these numbers, and they have many interesting properties. Have students add any two consecutive triangular numbers. This sum is always a perfect square. Ask students if they can find, geometrically, a property related to this, starting with the triangles pictured above. What needs to be done is to connect the points and make little triangles inside the large one. The first triangle contains one triangle only; the second contains 3 more or 4 altogether; the next contains 9 altogether; the next 16, and so on. Each of these numbers is, of course, a perfect square.





ACTIVITY 3: Redirect students' attention to squares, as follows:
Look at the following arrangement.



This is simply a playful arrangement of pairs of 1 × 1 squares, 2 × 2 squares, and 3 × 3 squares. Look, however, at the opening to the left. How many squares are missing? (8) They can be represented as 2 rows of 2 squares and 1 row of 4 squares, i.e., $2 \times 2 + 1 \times 4 = 8$

If we continue this idea by attaching a 4×4 square to the above drawing in a similar way, the "missing" squares can be described by: $3 \times 2 + 2 \times 4 + 1 \times 6 = 20$. This leads to the following pattern:

$1 \times 2 + 0 \times 4 = 2$

$2 \times 2 + 1 \times 4 = 8$

$3 \times 2 + 2 \times 4 + 1 \times 6 = 20$

$4 \times 2 + 3 \times 4 + 2 \times 6 + 1 \times 8 = 40$

$5 \times 2 + 4 \times 4 + 3 \times 6 + 2 \times 8 + 1 \times 10 = 70$

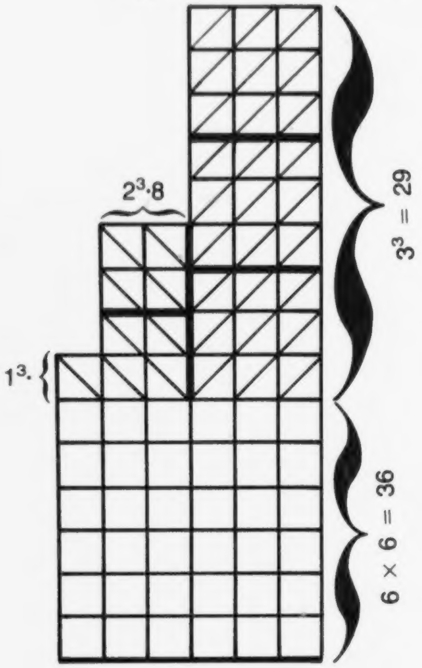
$6 \times 2 + 5 \times 4 + 4 \times 6 + 3 \times 8 + 2 \times 10\ 1 \times 12 = 112$

Notice that the arrangement of addends is symmetrical. Next, have students list the sums, find the difference between consecutive sums, factor the differences, and then find half of each difference. Some interesting results emerge:

Sum	Difference	Factored Difference	Difference/2
2	—	—	—
8	6	2×3	3
20	12	3×4	6
40	20	4×5	10
70	30	5×6	15
112	42	6×7	21
168	56	7×8	28

Ask students if the numbers in the last column look familiar (of course, they are the triangular numbers).

ACTIVITY 4: Lastly, let's look at the expression $6 = 1 + 2 + 3$ again, by way of introducing a suggested project. Make and explain the following arrangement.



Basically, what's being done here is that a total area of 36 is being re-arranged into three sub-areas, each sub-area being a cube of an integer. Elicit that the rectangle containing 8 squares is made up of two 2×2 squares, and the rectangle containing 27 squares is made up of three 3×3 squares. We can think of each square as a floor in a building, and so when we construct actual cubes, we can have one 1×1 one-story structure, one 2×2 two-story structures, and one 3×3 three-story structure. These ideas can apply to working on the following projects.

EXTENDED LEARNING: Construct a 1-unit, 2-unit, 3-unit and 4-unit cube.

1. Let's say you're designing a building or a space, but you have only 36 square feet. You can design a one-level 6 × 6 square space (or other 36 square foot rectangular space), or you can select the 1, 2, and 3 unit cubes (why?) and arrange them in an interesting configuration.
2. Same problem as #1, but with 100 square feet.
3. Same as above, but with 72 square feet.

These projects allow students to work geometrically and 3-dimensionally with the ideas of squares, cubes, areas, volumes, and patterns.

LATTICE MULTIPLICATION

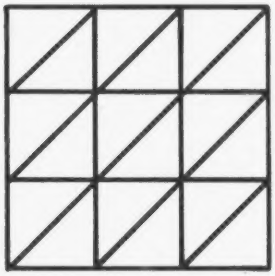
GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVE: The "galley" or lattice method of multiplication is a method used by ancient man. This method first appeared in one of the first published texts in Italy, the *Treviso Arithmetic* (1478).

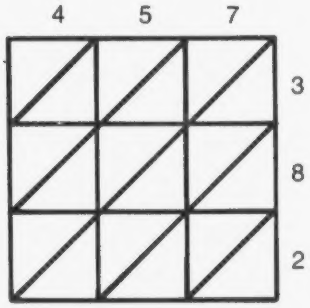
The student will solve various multiplication problems by constructing and using lattice method.

PROCEDURE: The teacher can use this method as another way for students to practice multiplication. Once the students are given the directions for the method, they can work independently.

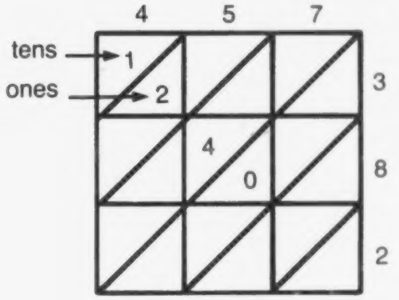
ACTIVITY 1: **Step 1.** Have students construct the lattice with 3 rows and 3 columns.



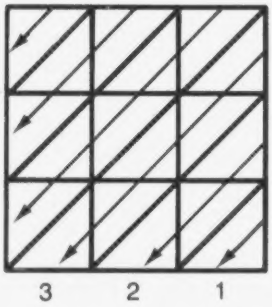
Step 2. Have students suggest possible placement for the digits 457 (in order from left to right at top) and 382 (in order from top to bottom at right).



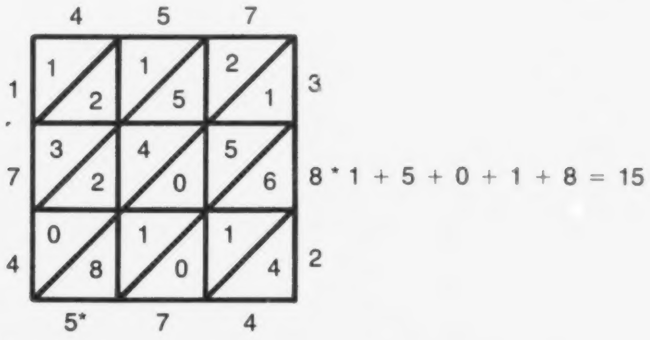
Step 3. Each product of a digit 457 and a digit 382 is called a partial product and is placed at the intersection of the column and row of digits. Tens digits are placed above the ones place.



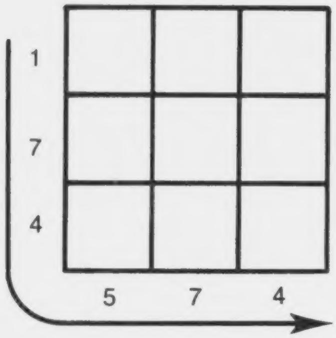
Step 4. After entering all partial products, the students add along the diagonals beginning in the lower right hand corner and carrying to the next diagonal sum.



Step 5. The teacher must demonstrate the carry-over to the next diagonal sum where necessary.



Step 6. The teacher will have a student read in direction of the arrow.



Solution 174, 574

ACTIVITY 2: The teacher will now provide an additional example.

ACTIVITY 3: Examples for more practice:

a) 693 × 236

b) 865 × 292

c) 587 × 627

d) 765 × 398

e) 581 × 237

EXTENDED LEARNING: Have students try other than three digit numbers.

MAGIC IN THREE-DIGIT NUMBERS

GRADE LEVEL/ COURSE: Algebra II

INTRODUCTION/ OBJECTIVE: The student will be able to prove algebraically why the "magic" of three-digit numbers works.

PROCEDURE: The teacher will allow students to demonstrate the "magic" of three-digit numbers, and then work with them to develop the algebraic proof.

ACTIVITY 1: Prior to the first step in the activity the teacher writes the number 1089 on a piece of paper.

Step 1. Have the students pick any three digit number in which the first and last digits are not the same.

Step 2. Reverse the digits of this number to form a new number.

Step 3. Subtract the smaller number from the larger number.

Step 4. Reverse the digits of this number to form a new number.

Step 5. Add the reversed number to the answer of the subtraction.

Note: All numbers must remain three-digit numbers. (i.e., use 0 as a placeholder)

Step 6. The teacher may freely move about the classroom while giving instructions. The piece of paper with the number 1089 written on it should remain on the teacher's desk at all times.

Step 7. Upon completion of the calculation, the teacher reveals the number on the piece of paper: 1089. This should correspond to all students final answer. In the Algebra II classroom, the proof of this "magic" can be presented after verification that the procedure will always produce the answer 1089.

ACTIVITY 2: **Proof:** Allow the students to attempt the proof using the traditional method of reversing the digits. The students proof will probably follow these steps:

$$\begin{array}{rl}
100x + 10y + z & \text{the original number}\\
-\,(100z + 10y + x) & \text{number with digits reversed}\\
\hline
100(x-z) + (z-x) & \text{answer}\\
+\ 100(z-x) + (x-z) & \text{number with digits reversed}\\
\hline
0 &
\end{array}$$

After the students discover that this procedure yields zero the teacher should explain that this occurred because this procedure did not allow for the fact that Z = Y which necessitates borrowing to complete the subtraction. Using a place value chart may help to visualize how this is done. The original number would look like this.

hundreds	tens	units
x	y	z

Borrowing from the 10's

to this:

hundreds	tens	units
x	y − 1	z + 10

Borrowing from the hundreds place changes it to this:

hundreds	tens	units
x − 1	y − 1 + 10	z + 10

Now the proof becomes:

$$\begin{array}{r} 100x + 10y + z \\ -(100z + 10y + x) \\ \hline \end{array} \Longrightarrow \begin{array}{rlll} & 100(x-1) & +10(y-1+10) & +1(z+10) \\ - & (100z & +10y & +\ x) \\ \hline & 100(x-1-z) & +10(y-1+10-y) & +1(z+10-x) \\ + & 100(z+10-x) & +10(y-1+10-y) & +1(x-1-z) \\ \hline \end{array}$$

$$100(x-1-z+z+10-x) + 10(y-1+10-y+y-1+10-y) + 1(z+10-x+x-1-z)$$

Which simplifies to:

$$\begin{array}{lll} 100(9) & +10(18) & +1(9) \\ 900 & +180 & +9 \\ & 1089 & \end{array}$$

ACTIVITY 3: A SIMPLIFIED VERSION

Step 1. Choose a three-digit number whose first and last digits are not the same.

Step 2. Reverse the digits.

Step 3. Subtract the smaller number from the larger number.

Note: A 9 will always be in the tens place. The digits in the hundreds place and units place will always add up to 9. Given *either* the first or last digits in the students answer, the teacher can name the entire answer to their problem. For example, if the first digit is a 3, then the answer is 396.

ACTIVITY 4: A MORE COMPLEX VERSION

Step 1. Choose *any* number whose first and last digits are different.

Step 2. Reverse only the first and last digits.

Step 3. Subtract the smaller from the larger.

Step 4. Reverse only the first and last digits in this number.

Step 5. Add these two numbers.

Note: The result will always begin with 10 and end with 89. The result has 9s in the middle such that the number of 9s is 3 less than the number of digits.

EXTENDED LEARNING:

$1089 \times 1 = 1089$

$1089 \times 2 = 2178$

$1089 \times 3 = 3267$

$1089 \times 4 = 4356$

•

•

•

$1089 \times 9 = 9801$

The students may be asked to discover the pattern formed by multiplying 1089 by each digit 1 through 9 and explain it algebraically.

REFERENCES:

Bakst, Aaron. *Mathematics Its Magic and Mastery.* New York: D. Van Nostrand Company, 1967.

Frankenstein, Marilyn. "Using Mathematical Magic to Reinforce Problem-Solving Methods". *Mathematics Teacher.* February 1984.

MAGIC SQUARES

GRADE LEVEL/ COURSE: 5–6

INTRODUCTION/ OBJECTIVE: Students will fill in the numbers in a magic square. A magic square is a square divided into equal parts (3 × 3, 4 × 4, 5 × 5, . . .) with numbers in each part. When added, these numbers should have the same sum vertically, horizontally, and diagonally. In ancient times these squares were thought to have magical properties.

PROCEDURE: Prepare 4 × 4 magic square form and 16 cards for each student and have them arrange their own magic square following the activity directions.

ACTIVITY:

Step 1. Give each student a 4 × 4 magic square and 16 cards that will fit into the the squares. Label 4 of the cards 1; 4 of the cards 2; 4 of the cards 3; and 4 of the cards 4.

Step 2. Instruct the students to arrange the cards on the magic square so that the sum of all rows, columns, and diagonals is 10.

4	1	3	2
2	3	1	4
1	4	2	3
3	2	4	1

EXTENDED LEARNING: The sum of any row, column, or diagonal of a magic square is called the magic constant.

1. Using numbers 1–9, make a 3 × 3 magic square that has a magic constant of 15.

4	9	2
3	5	7
8	1	6

2. Artist Albrecht Dürer made a famous magic square in 1514.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

All diagonals, rows, and columns add up to 34.

In this magic square the 4 numbers in the corners also add up to 34. The 4 numbers in the center add up to 34. Find the other numbers that add up to 34.

3. To determine the magic constant in a magic square, add all the numbers in the square (a 3 × 3 or 3rd order magic square would have numbers 1–9). $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$. Divide by 3. The answer to this division problem is the magic constant for a 3rd order magic square.

REFERENCE:

Bendick, Jeanne, *Take a Number.* New York: McGraw Hill, 1961.

Bezusha, Stanley. *Contemporary Motivated Mathematics Book 1.* Chestnut Hill, Mass.: Boston College Press, 1972.

MAGIC TRIANGLES

GRADE LEVEL/ COURSE: Algebra I

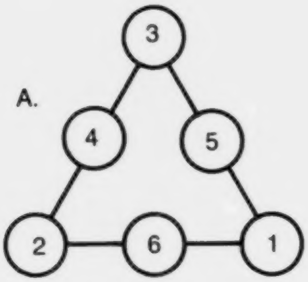
INTRODUCTION/ OBJECTIVE: Students will discover relationships and find patterns that exist in Magic Triangles.

PROCEDURE:

1. The teacher should introduce the basic concept of a magic triangle—by completing an example for the class.
2. The teacher should guide the students in the development of a chart to record the relationships.
3. The teacher should encourage the students to make generalizations.

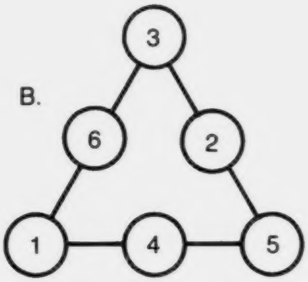
ACTIVITY 1: Challenge students to fill in the circles in triangle A using the numbers, 1, 2, 3, 4, 5, and 6 so that the sum along each side is 9. Each digit is to be used exactly once.

A.



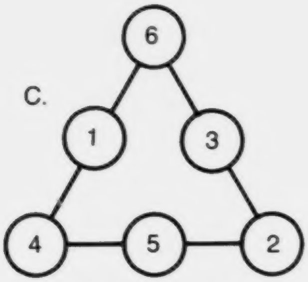
Allow sufficient time for the students to complete the magic triangle. Then ask the students to complete the following triangles using the same digits as before with the sum along each side as indicated below each diagram.

B.



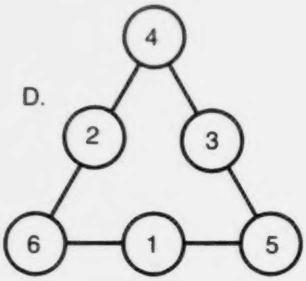
sum along each side is 10

C.



sum along each side is 11

D.



sum along each side is 12

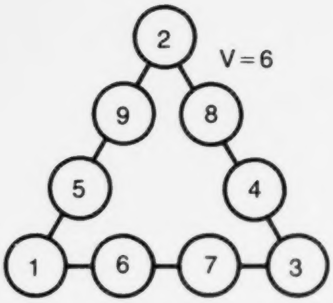
ACTIVITY 2: Using the above results have the students generate Chart I to establish a pattern showing the relationship among the side sum (s), the vertex sum (v) and the total sum of the digits (t). Explain that the total sum of the digits represents the sum of each side (3s) minus the sum of the vertices since the vertices are not to be counted twice in the total sum. Thus the following formula: $t = 3s - v$.

Chart I

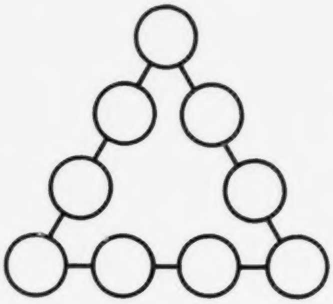
sum along each side (s)	9	10	11	12
sum of the vertices (v)	6	9	12	15
sum of the digits (t); $t = 3s - v$	21	21	21	21

ACTIVITY 3: Discuss the pattern formed as the sum of each side increases by one (the sum of the vertices increases by 3). Also verify that the sum of the digits is equal to $3s - v$.

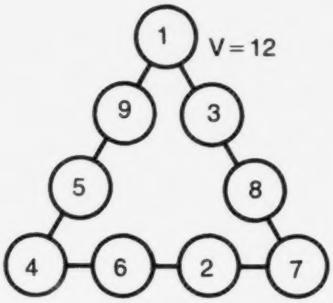
ACTIVITY 4: Now introduce magic triangles with 4 circles on each side. Using the numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, fill in the circles so that the sum along each side in each triangle is 17, 18, 19, 20, 21, 22, 23 respectively. Indicate that two of these triangles have no solutions ($S = 18$ and $S = 22$). Complete these triangles by first using the formula $t = 3s - v$ to determine the sum of the vertices, then fill in the vertices and complete the sides.



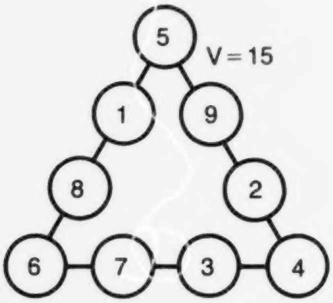
sum is 17



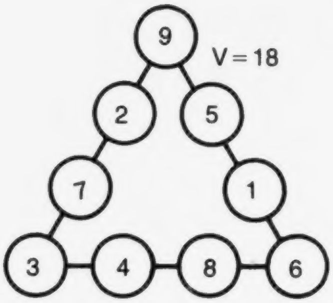
sum is 18



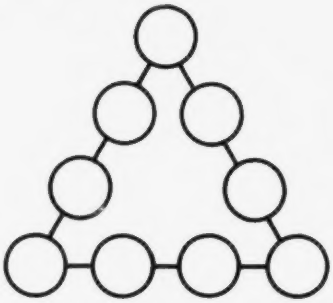
sum is 19



sum is 20

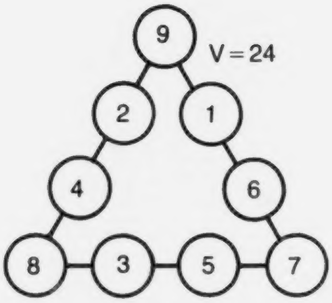


sum is 21



sum is 22

ACTIVITY 4: (Continued)



sum is 23

ACTIVITY 5: Now generate Chart II to verify that the sum of the digits 1 through 9 is the same as $t = 3s - v$ or 45.

Chart II

sum along each side (s)	17	18	19	20	21	22	23
sum of the vertices (v)	6		12	15	18		24
$3s - v$	45		45	45	45		45

ACTIVITY 6: Now introduce magic triangles with 5 circles on each side and use numbers 1 through 12 to fill in the circles so that the sum along each side in each triangle is 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, respectively.

Using information from the preceding magic triangle, complete another chart verifying the results of the relationship of the sum of the digits used and the formula $t = 3s - v$.

EXTENDING ACTIVITIES:

1. If there are 6 circles on each side of the magic triangle, how many triangles would be expected to be completed?
2. Examine the relationship between the number of circles (n) on each side of the magic triangle and the numbers of triangles to be expected to be completed.
3. Solve some magic triangles of 6 or more circles on each side.

REFERENCES:

Caldwell, Janet. "Magic Triangles". *Mathematics Teacher.* Vol. 71. January 1978.

Yates, Daniel S. "Magic Triangles and a Teachers Discovery". *Arithmetic Teacher,* Vol. 23. May 1976.

MATH TRICKS

GRADE LEVEL/ COURSE: Algebra I

INTRODUCTION/ OBJECTIVES:

1. Students will try to determine why 5 is always the answer to a given mathematical puzzle.
2. Students will try to determine someone's age, and tell them how much change (under a dollar) they have in their pocket.

Students need only basic skills of addition, subtraction, multiplication and division. Vocabulary should include variable and identity.

ACTIVITY 1:

Start with any number, e.g.,	15
Add the next larger number.	15 + 16 = 31
Add 9 to this sum.	31 + 9 = 40
Divide the result by 2.	40/2 = 20
Subtract the original number.	20 − 15 = 5

The answer is 5, and is always 5, no matter what number was chosen at first.

We need proof for this last statement. Using the variable N as a place holder is a good way to show this. If you follow the steps in order, you will always get an answer of 5 in the last step!

Here is the proof:

Start with a number.	N
Now add the next larger number.	$N + (N+1) = 2N + 1$
Add 9 to the sum.	$2N + 1 + 9 = 2N + 10$
Divide the result by 2.	$(2N + 10)/2 = N + 5$
Subtract the original number.	$N + 5 - N = 5$

Therefore, the answer is 5, and the proof is complete.

ACTIVITY 2:

This neat trick is one that will keep your student's interest! This trick will enable you to find a person's age and also tell the amount of change he is carrying at the same time.

Ask the person to write his age on a piece of paper. Suppose that his age is 15.

Multiply by 4	15 (4) = 60
Add 10	60 + 10 = 70
Multiply by 25	25 (70) = 1750
Subtract the number of days in a year (not a leap year)	1750 − 365 = 1385
Add the change in your pocket that is less than one dollar (say 38 cents)	1385 + 38 = 1423
Add 115	1423 + 115 = 1538

When the person gives you this answer, you can immediately tell him that his age is 15 and the change he has in his pocket is 38 cents because the first two digits in the answer are 15 and the last two are 38—the age and change respectively.

ACTIVITY 3: This trick can also be explained by using algebra.

Start with age	A
Multiply by 4	4A
Add 10	4A + 10
Multiply by 25	25(4A + 10) = 100A + 250
Subtract the number of days in a year, 365	100A + 250 − 365 = 100A − 115
Add the change less than one dollar, C, in his pocket	100A − 115 + C
Add 115	100A − 115 + C + 115 = 100A + C

100A places the ones digit of A in the hundreds place of the final answer. C then occupies the ones place or the ones and tens places.

If the change is more than a dollar, the numeral in the hundreds place will be determined partly by the change and partly by the age; therefore you will not be able to separate the two.

EXTENDED LEARNING: Have the students try these two "tricks" using other numbers, ages and amounts of change.

REFERENCES:

Glenn, William H. and Johnson, Donovan A. *Fun With Mathematics.* Lakeland Fla.: Webster Publishing Company, 1961.

MEASUREMENT AND CONSTRUCTION

GRADE LEVEL/ COURSE: 4–6

MATERIALS: An art center will be an important part of the experience. The following materials will be needed in the center: a collection of floor plans clipped from magazines and newspapers, corrugated cardboard, wallpaper scraps, carpet scraps, glue, scissors, rulers, drawing paper, tape, tempera paints, brushes.

INTRODUCTION/ OBJECTIVE: The student will use linear measurement to construct a scale model house of cardboard.

PROCEDURE: This activity offers students who are proficient in linear measurement an opportunity to make practical application of those skills as they design and build their ideal house from corrugated cardboard. Either metric or customary measurement may be used. The project can be best managed when it is used only with highly proficient students.

ACTIVITY: The following directions should be duplicated and given to students:

Step 1. Study the floor plans in the art center. Decide the floor plan of your dream house. You may use one you like from the center or design your own. Draw the floor plan to a scale which will fit a piece of 18″ by 24″ drawing paper.

Step 2. Use cardboard and tape to construct the inside and outside walls of your house, using the same scale as your floor plan.

Step 3. Cut windows and doors also to scale.

Step 4. Paint the outside of the house as you would like it to be.

Step 5. Calculate the area of each room. Using wallpaper scraps to paper the walls and carpet scraps to carpet the floors.

EXTENDED LEARNING: Draw a floor plan for your home or your school.

MIND READING TRICKS

GRADE LEVEL/ COURSE: Algebra I

INTRODUCTION/ OBJECTIVE: Given a mind reading trick, the student will determine an algebraic solution.

PROCEDURE: Introduce each "trick" by acting as the "mind reader" and have students follow the directions.

After predicting each solution have the students generate an algebraic justification. The teacher may need to suggest that students analyze each operation.

ACTIVITY 1:

Step 1. Have the students think of a number; multiply it by 5 and add 6; multiply by 4 and add 9; multiply by 5 and state the result.

Step 2. The "mind reader" subtracts 165 and divides by 100 to determine the original number.

Step 3. When each operation is analyzed the operations are $5a$, $5a + 6$, $20a + 24$, $20a + 33$, and finally $100a + 165$.

ACTIVITY 2:

Step 1. Have each student multiply his/her age by 2; add 5; multiply the result by 50; add the amount of change in your pocket (less than $1.00); subtract 365 (the number of days in a year); and state the result.

Step 2. The "mind reader" adds 115 to the result and the first two digits will be the person's age and the last two digits will be the amount of change.

Step 3. When each operation is analyzed let a = age and b = pocket change. The operations are: $2a$, $2a + 5$, $100a + 250$, $100a + b + 250$, and finally $100a + b - 115$. When 115 is added to the last number the answer is $100a + b$.

ACTIVITY 3:

Step 1. Have the student think of a three digit number in which the first and last digits are different; reverse the digits and subtract the smaller from the larger; and give the first digit of the result.

Step 2. The "mind reader" knows that the middle digit is always 9 and the sum of the first and last digits is nine. He can therefore state the result.

Step 3. Representing the original number as $100a + 10b + c$; reversing the digits gives $100c + 10b + a$. The difference of these is $99(a - c)$. In multiples of 99 the middle digit is always 9 and the sum of the first and last digits is also 9 (Ex. 99, 198, 297, 396, 495 . . .).

ACTIVITY 4:

Step 1. Have the students think of a three digit number in which the first and last digits differ by more than one; reverse the digits; subtract the smaller number from the larger; reverse the digits; add this number to the answer of the subtraction.

Step 2. The "mind reader" knows the result is 1089.

Step 3. Representing the original number as $100a + 10b + c$ the reverse is $100c + 10b + a$. The subtraction problem is written:

$$\begin{array}{r} 100\,a + 10\,b + c \\ \underline{-(100\,c + 10\,b + a)} \end{array}$$

which equals: (To perform the subtraction it is necessary to "borrow" in the ones and the tens place)

$$\begin{array}{rlll} & 100(a-1) & +\ 10(b-1+10) & +\ (c+10) \\ & -(100\,c & +\ 10\,b & +\ \ a\ \ ) \\ \hline 100 & (a-1-c) & +\ 10(b-1+10-b) & +\ (c+10-a) \end{array}$$

Reversing digits and adding the result is:

$$\begin{array}{rlll} & 100\,(a-1-c) & +\ 10(9) & +\ (c+10-a) \\ + & 100\,(10+c-a) & +\ 10(9) & +\ (a-1-c) \\ \hline & 900 & +\ 90+90 & +\ 9 = 1089 \end{array}$$

REFERENCES: Posamentier, Alfred S., and Sheridan, Gordon. *Math Motivators! Investigations in Pre-Algebra.* Reading Mass.: Addison-Wesley Publishing Company, 1982.

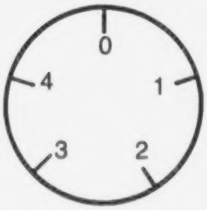
Northrop, Eugene P. *Riddle in Mathematics.* New York: D. Van Nostrand Company, 1944.

MODULO ARITHMETIC (FIRST ORDER OPERATION)

GRADE LEVEL/ COURSE: Algebra I

MATERIALS: Modular 5 clocks with moveable hands

INTRODUCTION/ OBJECTIVE: Students will investigate relationships in a modular system. A system of arithmetic based on a clock is modulo arithmetic. Here a five-hour clock will be used as a starter. The clock will be numbered 0, 1, 2, 3, and 4.



—Addition on this clock may be performed by using a clockwise direction. 3 + 4 indicates we start at 0 and move 3 units + 4 more = 2.

—Multiplication is defined as repeated addition. 3 × 4 is equivalent to 4 + 4 + 4 which is 4 + 4 = 3 and 3 + 4 = 2; therefore 3 × 4 = 2.

—Division: "$\frac{a}{b} = x$" is equivalent to "$a = bx$"

The system called an arithmetic modulo 5 is written $3 + 4 = 2 (\text{mod} 5)$.

The students will solve first-order equations in one variable using modular arithmetic in modular 5 system.

PROCEDURE: The first 6 activities are teacher directed activities. Students must be able to solve simple first-order equations before completing this activity such as $\frac{3}{x} = 2$ and $x + 3 = 5$. The remaining activities can be independent activities. These activities can be with large or small groups.

ACTIVITY 1: Have the students use their model of modular 5 clock to add 3 + 3 to get 1. Other examples: 4 + 3 = 2, 4 + 4 = 3, etc.

ACTIVITY 2: Have the students use their model of modular 5 clock to multiply 3 × 4. This is 4 + 4 = 3 and 3 + 4 = 2.

ACTIVITY 3: Have students notice that 2 numbers are congruent modulo 5 if and only if they differ by a multiple of 5. Thus, 3, 8, 13, and 18 are all congruent to each other modulo 5.

ACTIVITY 4: Have students discover that numbers 0, 1, 2, 3, 4 represent all possible remainders of any integer divided by 5.

ACTIVITY 5: Have students use their modular 5 clock to solve first order operation $2 - 3 = x$ which is equivalent to $2 = 3 + x$. By moving from 0 to three and then to 2, the students can see they moved 4 units from 3 to 2; therefore $3 + 4 = 2$ or $x = 4$.

ACTIVITY 6: Have students use their clocks to find x for $\frac{3}{4} = x$ which is equivalent to $3 = 4x$.

Move to 4 from 0 and now to 3. How many more 4s did you move? (2) Therefore $3 = 4(2)$ or $4 + 4 = 3$, $x = 2$.

ACTIVITY 7: Have students solve the following equations for x using modulo 5 system:

$$\begin{aligned} 2 + x &\equiv 1 \pmod{5} \\ x + 3 &\equiv 2 \pmod{5} \\ 4x &\equiv 1 \pmod{5} \\ \frac{4}{3} &\equiv x \pmod{5} \\ 3x &\equiv 1 \pmod{5} \\ 3 - x &\equiv 4 \pmod{5} \end{aligned}$$

EXTENDED LEARNING: Have the students develop their own modular system and use that system to solve the above equations.

REFERENCES:

Posamentier, Alfred S. and Stepelman, Jay. *Teaching Secondary School Mathematics.* Columbus, Ohio: Merrill Publications, 1981.

THE MÖBIUS STRIP

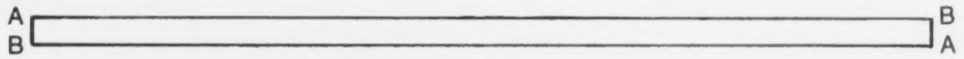
GRADE LEVEL/ COURSE: 6–12

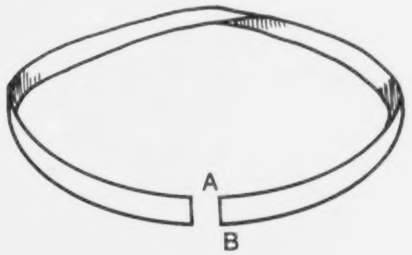
INTRODUCTION/ OBJECTIVE: This activity is designed to allow students to explore a topic from modern mathematics. The student will discover some topological properties through experimentation.

The Möbius strip is named after one of the pioneers in topology, August Ferdinand Möbius (1790–1868), a German mathematician and astronomer. The Möbius strip contains a half-twist. It is constructed by twisting one end of a rectangular strip through an angle of 180° before taping it to the other end. Because of this half-twist, the Möbius strip has only one surface and one edge. This causes it to have a number of strange properties that are easily discovered by experiment.

B. F. Goodrich Co. has patented a conveyor belt in the shape of a Möbius strip. It lasts longer because both sides are actually one and receive equal wear. The Möbius strip has been put to use in such areas as the design of electronic resistors, tape cartridges and aerobatic tricks performed by free-style skiers.

MATERIALS: Several sheets of graph paper, ruler, scissors, tape.





PROCEDURE: The teacher will instruct the entire class in this activity and lead the students to discovery through class discussion.

ACTIVITY 1: Have the students cut two strips, each as long as possible and 1 inch wide from graph paper.

Take one of the strips, make it into a loop and turn one end over before taping the two ends together as shown in the diagram above. (You now have a Möbius strip.)

Do the same with the other strip.

ACTIVITY 2: Place your pencil midway between the edges of one of the strips and draw a line down its center, continuing until you return to the point at which you started. Cut the band along the line you have drawn. What is strange about the result? (You will have one large band but with the normal two surfaces.)

ACTIVITY 3: Place your pencil midway between the edges of the resulting strip and draw a line down its center, until you return to the point at which you started. Cut the strip along the line drawn lengthwise down its center. What is the result? (Two two-sided bands linked within each other, impossible to separate.)

ACTIVITY 4: Cut the other Möbius strip that you made lengthwise along a line that is about ⅓ of the way across its surface. You will make two trips around the band, but only one cut. What is the result? (The result of this cut will be two strips intertwined: one will be a new Möbius strip, the other a two-surfaced figure.)

EXTENDED LEARNING: Have student experiment with strips the same size as those above. Follow Activities 1 to 4 above but put two half-twists in the bands. Compare the results. Then put 3 half-twists in two of the bands following the same Activities 1 to 4. Design a chart to record the experiments. Look for patterns which might lead to a conjecture. (A band with an odd number of half-twists in it has one side. When cut, it produces two interlocking bands, the longer band has a knot in it and the shorter band passes through the knot. A band with an even number of half-twists in it has two sides. When cut, it produces two interlocking bands.)

REFERENCES:

Barr, Steven. *Experiments in Topology.* New York: Simon and Schuster, 1937.

Jacobs, Harold R. *Mathematics: A Human Endeavor.* New York: W. H. Freeman and Company, 1982.

NETWORKS

GRADE LEVEL/ COURSE: Geometry

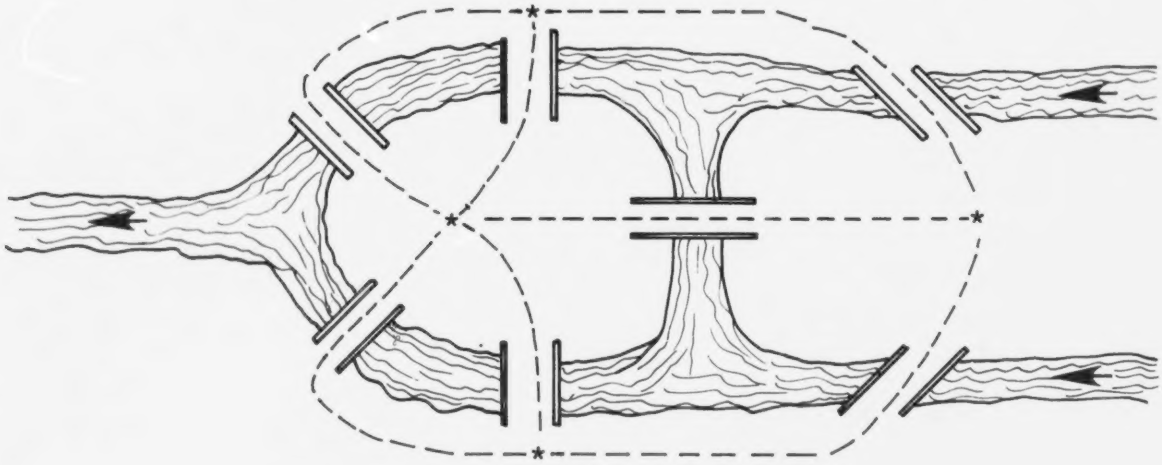
INTRODUCTION/ OBJECTIVE: This activity is designed to encourage students to use critical analysis to determine whether or not a problem can be solved.

PROCEDURE: Present each network and lead students through a step by step procedure until they can formulate a general theorem.

ACTIVITY 1:

Network #1

The student will determine why the Königsberg Bridge problem is not solvable.



*odd nodes

Königsberg (present name: Kaliningrad), a city in Prussia which is now a part of Russia, had seven bridges crossing a river and its tributaries as shown in the figure. The problem was that not one person could cross all seven bridges without crossing at least one bridge more than once. It was not until the 17th century when Leonhard Euler, a Swiss mathematician, determined the problem was a network problem which he proved could not be solved.

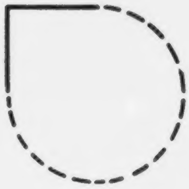
Definitions: **Node**—any point where two or more line segments meet.

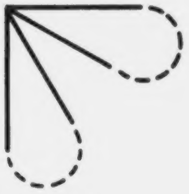
Odd node—any point where any odd number of line segments meet.

Even node—any point where an even number of line segments meet.

Step 1. The student should read through each step of the procedure until he/she can achieve the end result; i.e., formulate a general theorem.

Step 2. Sketch several even nodes. Assume that each node is part of a figure that can be retraced in one stroke.


2 paths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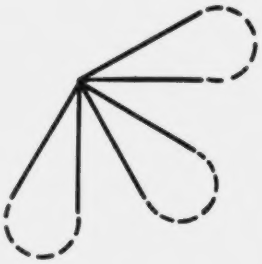
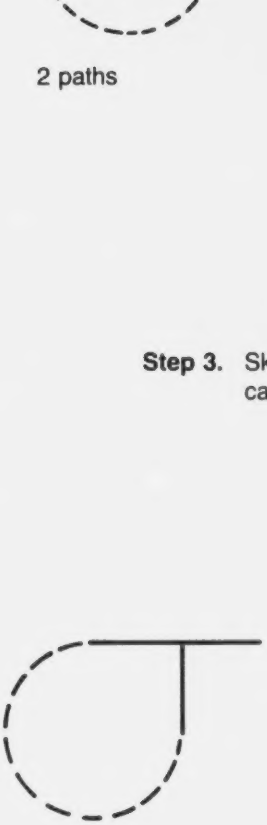
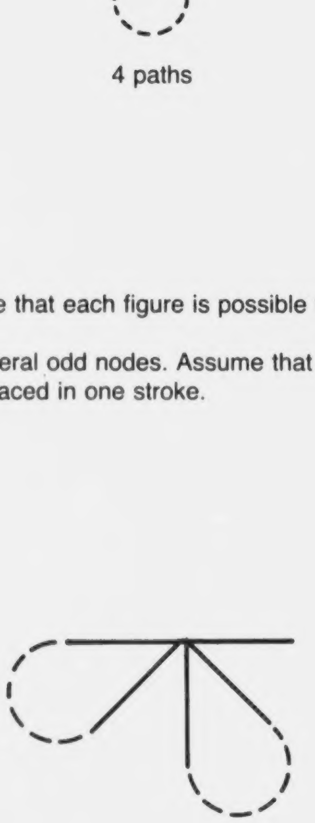
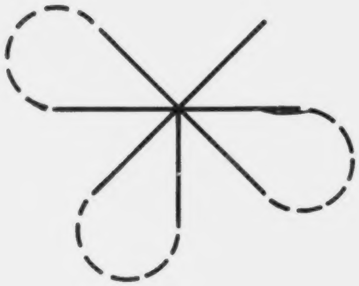

6 paths

Observe that each figure is possible to complete with only one stroke.

Step 3. Sketch several odd nodes. Assume that each node is part of a figure that can be retraced in one stroke.

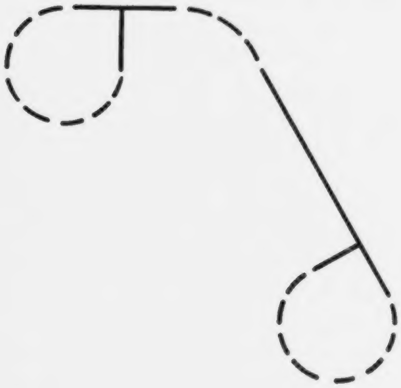

3 paths

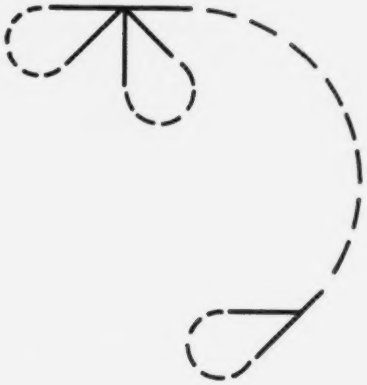

5 paths

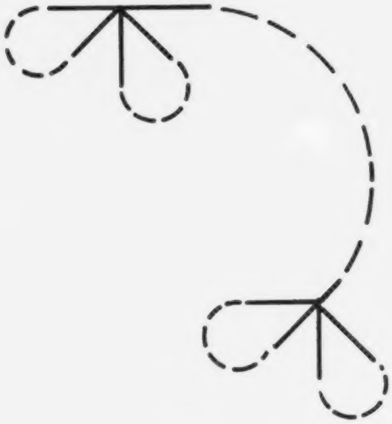

7 paths

Observe that each figure is possible to complete with one stroke only if started on the node.

Step 4. Sketch several figures with 2 odd nodes.







Observe that each figure with two odd nodes can be completed with only one stroke and that the figure can be retraced only if the pencil starts on one of the odd nodes and concludes on the other odd node.

Step 5. Sketch several figures with three odd nodes.

Observe that none of the figures with three odd nodes can be retraced with one stroke.

Step 6. Conclude from these results the general theorem:

A network problem can be solved only if there are two odd nodes in the path.

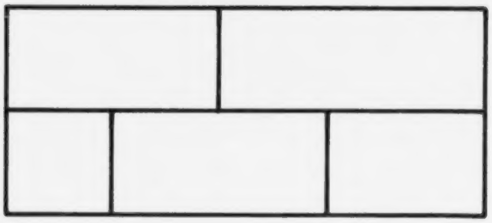
ACTIVITY 2:

Network #2

The student will determine whether or not the classic line drawing problem is possible, i.e., can the figure shown below be retraced with one stroke of the pencil?

The student will develop a general theorem stating how many strokes of the pencil are necessary to retrace any figure.

The student is asked to retrace the figure shown below with as few strokes of the pencil as possible. One stroke is defined as retracing a portion of the figure, without removing the pencil from the paper or going over a line twice.



Step 1. The student should read through each step of the procedure until he or she can achieve the end result; i.e., formulate a general theorem.

Step 2. Observe also that this second network problem adds the restriction that the figure must be a closed figure.

Step 3. Sketch the figure.

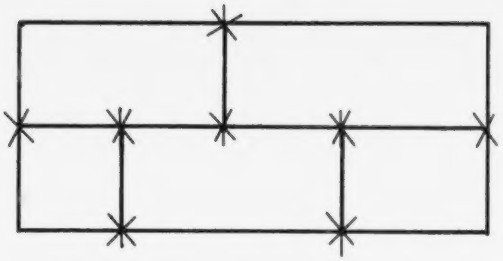
Step 4. By trial and error, retrace the figure with as few strokes of the pencil as possible, i.e., attempt 1, 2, 3, . . . strokes until a 'success' is obtained.

Step 5. After discovering that four strokes are necessary for a 'success', analyze your sketches to determine where the pencil had to be placed at the beginning and conclusion of each successful stroke.

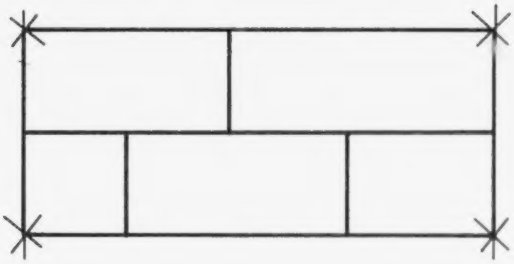
Observe that the pencil had to start at an odd node.

Observe that the pencil had to conclude at an odd node.

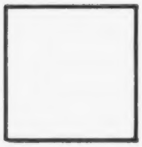
Observe that there are eight odd nodes in the figure.



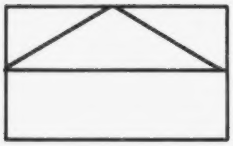
Observe that there are four even nodes in the figure below.



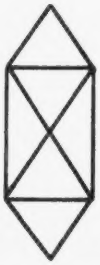
Step 6. Sketch several simple figures with only even nodes.



Four even nodes



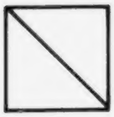
Seven even nodes



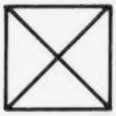
Seven even nodes.

Observe that each figure can be traced with one stroke, and that there is no restriction on where the pencil is to start on these figures.

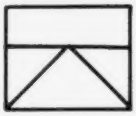
Step 7. Sketch several figures with odd and even nodes. Begin with a figure having only two odd nodes, then four odd nodes, etc.



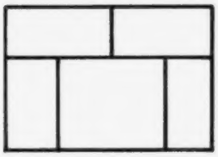
Two odd nodes



Four odd nodes



Six odd nodes



Eight odd nodes

(a) Observe that the figure with two odd nodes can be retraced with only one pencil stroke if the pencil starts at one of the odd nodes and concludes at the other.

(b) Observe that the figure with four odd nodes can be retraced with a minimum of two strokes.

(c) Observe that the figure with six odd nodes can be retraced with a minimum of three strokes.

(d) Observe that the figure with eight odd nodes can be retraced with a minimum of four strokes.

Step 8. Conclude from these results the following general theorem:

> Half as many strokes as there are odd nodes
> will be necessary to retrace the complete figure.

REFERENCES:

Schuh, Fred. *The Mastery Book of Mathematical Recreations.* New York: Dover Publications, 1968.

Bell, E. T. *Men of Mathematics.* New York: Simon and Schuster, 1937.

James, Glenn and James, Robert. *Mathematics Dictionary.* New York: Van Nos Reinhold, 1976.

NUMBER ODDITIES

GRADE LEVEL/ COURSE: 4–6, 7, 8

INTRODUCTION/ OBJECTIVE: The students will solve number oddities, and apply the solution to other similar problems.

PROCEDURE: The teacher may use any of these activities with large or small groups to stimulate critical thinking, listening, mental and/or calculator math, and to have fun with math.

ACTIVITY 1: The process which follows can be repeated until, at some stage, the sum reads exactly the same from left to right and from right to left.

Step 1. Choose any two digit number.

Step 3. Reverse the digits.

Step 3. Add the two numbers.

Step 3. Do the same to the sum thus obtained.

Comment: This activity, although brief, may be used to stimulate interest in number oddities.

ACTIVITY 2: This is an exercise whereby the teacher and students will comprise a list of five 4-digit numbers to be added. Upon completion of identification of the 5th number, the teacher will immediately write the sum. (i.e., "lightning addition")

Step 1. The students choose the 1st and 2nd numbers.

Step 2. The teacher chooses the 3rd number. (When added to student's second = 9999)

Step 3. The students choose the 4th number.

Step 4. The teacher chooses the 5th number. (When added to student's 4th number = 9999)

Step 5. The teacher then draws a horizontal line to separate the problem and the answer and then immediately writes the answer to the addition problem.

Comment: By generating several examples the students will be given an opportunity to evaluate the hypothesis.

Solution: The teacher's third number is not randomly selected but must total 9999 with each student's second number (which is one less than 10,000) and the student's 4th and the teacher's 5th numbers must have the sum of 9999 (which is one less than 10,000). The final answer will always be 20,000 + (first number) − 2.

Variations: This activity can be simplified by using 3-digit numbers or can be made more complex by using numbers with more than 4 digits. The teacher may also vary the presentation by selecting the 2nd and 4th numbers, or the

2nd and 5th numbers, or the 4th and 5th numbers. The students *must* choose the 1st number.

ACTIVITY 3: For this activity the classroom may be divided into groups. Each group must be provided with a pair of dice.

Step 1. Roll the dice.

Step 2. Find these products:

a. the top two numbers

b. the bottom two numbers

c. the top number of the 1st die and the bottom number of the 2nd die

d. the top number of the 2nd die and the bottom number of the 1st die

Step 3. Add these products.

Comment: The answer to the calculations will always be 49. The observation needs to be made that the opposite faces of a die will always total 7 when added.

ACTIVITY 4: Calculators may be used as the teacher sees fit.

Step 1. Enter the number of your birth month.

Step 2. Multiply it by 10.5.

Step 3. Add 20 to this product.

Step 4. Multiply this sum by 10.

Step 5. From this product subtract the product of 5 and your birth month.

Step 6. Add 165 to this difference.

Step 7. Add your age to this sum.

At this point the teacher asks a student to volunteer the answer to the calculations. By subtracting 365 from the answer, the teacher is able to announce the student's birth month and age.

Solution: The number represented by the tens and units digits is the person's age and remaining digits represents his birth month.

ACTIVITY 5: Have the students:

Step 1. Select any number.

Step 2. Double it.

Step 3. Add 4 to this product.

Step 4. Multiply this sum by 5.

Step 5. Add 12 to this product.

Step 6. Multiply this sum by 10.

Step 7. Subtract 320 from this product.

Step 8. Divide this difference by 100.

Note: The quotient will always be the original number selected by the student.

REFERENCES:

Bakst, Aaron. *Mathematics Its Magic and Mastery.* New York: D. Van Nostrand Company, 1967.

Frankenstein, Marilyn. "Using Mathematical Magic to Reinforce Problem-Solving Methods". *Mathematics Teacher.* February 1984.

Garman, Brian. "The Calculator, Math Magic and Algebra" September 1984.

NUMBER THEORY PROBLEMS INVOLVING CONGRUENCE AND DIVISIBILITY

GRADE LEVEL/ COURSE: Algebra II and above

INTRODUCTION/ OBJECTIVE: This activity is designed to help students understand and use the theorems of congruence and divisibility to solve number theory problems.

PROCEDURE:

1. The teacher should introduce the concept of divisibility and the basic theorems of congruence.
2. The teacher should guide the students in the solution of sample problems.

ACTIVITIES: Before anything really worthwhile can be discussed in number theory, several basic ideas and theorems must briefly be discussed. Probably the most basic is congruence of integers. Two integers are said to be congruent to each other with respect to a third non-zero integer if and only if they have the same positive remainder after division by that third number. Some examples are: 5 and 9 are congruent with respect to 2; 7 and -2 are congruent with respect to 9, etc. Another way to describe this relationship is to say that either of the differences of the two integers is divisible by the third. That is, $5-9$ and $9-5$ are divisible by 2 and $7-(-2)$ and $-2-7$ are divisible by 9. Gauss developed the following notation: if $a-b$ is divisible by m, we write $a \equiv b \pmod{m}$ (read "a is congruent to b modulo m). So our previous examples may be written as $5 \equiv 9 \pmod{2}$ and $7 \equiv -2 \pmod{9}$. The notation seems to indicate an analogy between congruence and equality, and, in fact, rules for manipulating congruences are about the same as those for equations.

Two basic theorems are:

1. If $a \equiv b$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$ (a transitive property).

The proof is simple: since $a-b$ and $b-c$ are divisible by m, $(a-b) + (b-c)$ is divisible by m. But $(a-b) + (b-c) = a-c$, so $a-c$ is divisible by m. That is, $a \equiv c \pmod{m}$.

2. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

(i) $(a+c) \equiv (b+d) \pmod{m}$

(ii) $(a-c) \equiv (b-d) \pmod{m}$

(iii) $ac \equiv bd \pmod{m}$

Each of the three above results can be proved in a similar manner to the first theorem. Also, using the results repeatedly, we obtain one more additional fact: if $a \equiv b \pmod{m}$ and k is a positive integer, than $a^k \equiv b^k \pmod{m}$.

For example, since $44 \equiv 26 \pmod 6$ and $3 \equiv 9 \pmod 6$, the following hold also:

$47 \equiv 35 \pmod 6$ (from 2i)
$41 \equiv 17 \pmod 6$ (from 2ii)
$132 \equiv 234 \pmod 6$ (from 2iii)
$3^2 \equiv 9^2 \pmod 6$ (additional result)

We should also note that $a \equiv 0 \pmod m$ means m divides a.

At this point, we can solve many interesting problems that would have been very difficult, if not impossible, a short time ago. Some examples are:

1. Find the remainder when 7^{10} is divided by 51. We want a number x such that
   $0 \leq x \leq 51$ and $x \equiv 7^{10} \pmod{51}$
   $7^{10} \equiv (7^2)^5 \equiv (49)^5$ and $49 \equiv -2 \pmod{51}$
   So $7^{10} \equiv (49)^5 \equiv (-2)^5 \equiv -32 \equiv 19 \pmod{51}$. Therefore $x = 19$.

2. Show that $3^{10} - 1$ is a multiple of 8.
   $3^{10} \equiv (3^2)^5 \equiv 9^5 \equiv 1^5 \equiv 1 \pmod 8$ (since $9 \equiv 1 \pmod 8$)
   Therefore $3^{10} - 1 \equiv 0 \pmod 8$ or $3^{10} - 1$ is divisible by 8 (i.e., is a multiple of 8).

3. Prove that $2^{11} - 1$ has the factor 23.
   $2^{11} \equiv 2 \cdot 2^{10} \equiv 2 \cdot (2^5)^2 \equiv 2 \cdot 32^2 \equiv 2 \cdot 9^2 \equiv 162 \equiv 1 \pmod{23}$.
   So $2^{11} - 1 \equiv 0 \pmod{23}$ and thus 23 is a factor of $2^{11} - 1$.

4. Prove that the sum of any positive integer and its square is even. Let k be a positive integer. We must show that $k^2 + k$ is even. That is, we must show that $k^2 + k \equiv 0 \pmod 2$. This proof turns out to be surprisingly easy: $k^2 + k \equiv k(k+1)$. Exactly one of k and $k+1$ is even. Therefore either 2 divides k or 2 divides $k+1$. In either case, 2 divides $k(k+1)$. That is, 2 divides $k^2 + k$.

At this point, the students are ready for a further important definition and theorem. If a and b are integers, then the greatest common divisor of a and b, written gcd(a,b) is the largest positive integer which divides both a and b. For example, $\gcd(5,10) = 5$, $\gcd(36,90) = 18$, etc. If $\gcd(a,b) = 1$, we say a and b are "relatively prime".

An important result is the following:

Suppose $\gcd(a,m) = 1$ and $ab \equiv ac \pmod m$.
So m divides $ab - ac$ and $ab - ac = mt$ where t is an integer.
$ab - ac \equiv a(b-c)$; so a divides $ab - ac$, so a divides mt.
But, since $\gcd(a,m) \equiv 1$, a must divide t.
So $b - c \equiv m\,(t/a)$ and t/a is an integer.
Thus $b \equiv c \pmod m$.
This is actually a very important result.

If $\gcd(a,m) \equiv 1$ and if $ab \equiv ac \pmod m$, then $b \equiv c \pmod m$. This means we can "cancel" numbers which are relatively prime to the modulus. Now, we are ready for one of the nicer theorems of number theory, Fermat's Theorem: If p is prime, and p does not divide a, then $a^{p-1} \equiv 1 \pmod p$.

The proof is somewhat more involved than the previous one and may possibly be omitted, depending on the level of the students in the class.

Consider the numbers a, 2a, 3a, . . ., $(p-1)a$
We want to show each of these numbers is congruent to exactly one of
1, 2, 3, . . ., $p-1$

Note: na is congruent to at least one of 1, 2, . . ., $p-1$ for $n=1, 2, \ldots, p-1$ since $na \not\equiv 0 \pmod{p}$ (remember p does not divide a).

Suppose ma and na with $m < n$ are congruent to the same number,
i.e. $na \equiv ma \pmod{p}$.
$\gcd(a,p)=1$ since p does not divide a.
From the previous result (above), since $\gcd(a,p)=1$ and $na \equiv ma \pmod{p}$, we can cancel the a to get $n \equiv m \pmod{p}$.
But, n and m are distinct numbers between 1 and $p-1$; so, $0 < n-m < p$ and p does not divide $n-m$.
This contradicts $n \equiv m \pmod{p}$.
So $(a)\,(2a) \ldots (p-1)a \equiv (1)\,(2) \ldots (p-1) \pmod{p}$.
Or $a\ p-1[1 \cdot 2 \cdot \ldots \cdot (p-1)] \equiv 1 \cdot 2 \cdot \ldots \cdot (p-1) \pmod{p}$.
Again, by our previous result, we may cancel.
So $a^{p-1} \equiv 1 \pmod{p}$.

With these results, many interesting and useful problems can be done. Several brief examples are:

5. Find the remainder when 10^{49} is divided by 7.
   $10^{49} \equiv 10 \cdot 10^{48} = 10(10^6)^8 \equiv 10\,(1^8)$ (by Fermat's Theorem)
   And $10 \cdot 1^8 = 10 \equiv 3 \pmod{7}$.

6. Prove that $9^{50}-4$ is divisible by 7.
   $9^{50}-4 \equiv 2^{50}-4 = 2^2 \cdot 2^{48}-4 = 2^2(2^6)^8-4 \equiv 2^2 \cdot 1^8-4$ (by Fermat's Theorem)
   And $2^2 \cdot 1^8-4 \equiv 0 \pmod{7}$.

7. Prove that if p is a prime and if a is any integer, then $a^p \equiv a \pmod{p}$. If p does not divide a, then by Fermat's Theorem: $a^{p-1} \equiv 1 \pmod{p}$. Multiplying by a: $a^p \equiv a \pmod{p}$, if p divides a, then p divide a^p so $a^p \equiv a \pmod{p}$ since both are congruent to $0 \pmod{p}$.

EXTENDED LEARNING:

1. Find the remainder when 21^{10} is divided by 51.
2. If k is any odd number, show $k^2 \equiv 1 \pmod{8}$ (Hint: write k as $2t+1$).
3. Prove that $5^{36}-1$ is divisible by 13.

REFERENCES:

National Council of Teachers of Mathematics. Enrichment Mathematics for High School. 28th Yearbook. Reston, Virginia: NCTM, 1967.

PASCAL'S TRIANGLE

GRADE LEVEL/ COURSE: 7–8, Algebra I

INTRODUCTION/ OBJECTIVE: This is designed to as a discovery activity for the student to develop an understanding of Pascal's triangle and its use as a learning tool.

HISTORY: Pascal's triangle is an array of natural numbers that appears in mathematical works as early as 1100 A.D. in China and Persia. It appeared in Western Europe somewhat later and its properties were explored by Blaise Pascal in "Traite du Triangle Arithmetique" which was published in 1665.

ACTIVITY: To properly discuss the Pascal triangle it is necessary to know how the rows and entries are named. The row numbers begin with zero and the zeroth row has one entry which is one. The terms in each row begin with the zeroth term, first term, second term etc. The terms in row 2 would be the zeroth term 1, the first term 2, and third term 1.

Row #		Sum
0	1	1
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512
10	1 10 45 120 210 252 210 120 45 10 1	1024

(Underlined numbers should be omitted for students)

1. Complete the above triangle using the pattern established in rows 0 through 4.
2. Add the numbers across each row.

Questions

1. How is each entry generated?

(Each number is added to the number next to it and the sum goes between the two numbers on the row below. One appears as the zeroth entry on each row as one plus nothing is one.)

2. What property or relationship do you observe in the row sums?

(Each sum is 2^n where n is the row number.)

$2^0 = 1$; $2^1 = 2$; $2^2 = 4$ etc.

As well as rows, the Pascal triangle has diagonals which are named in the same manner as the terms. Diagonals begin with the zeroth diagonal followed by the first diagonal, second diagonal, etc.

3. What number appears in the zeroth diagonal? (1)

4. What whole number does not appear in the triangle itself? (0)

5. What pattern appears in the first diagonal?

(The natural numbers in order)

6. What relationship can be observed between the first and second diagonals?

(The sum of the first n natural numbers where n is the row number of the row above the second diagonal term)

$1+2=3 \quad 1+2+3+4=6 \quad 1+2+3+4+5=15 \quad 1+2+3+4+5+6+7+8+9=45)$

EXTENDED LEARNING: The Pascal triangle can provide a pattern for the coefficients in a binomial expansion $(x+y)^n$ where n is the row number from the triangle. Fill in the coefficients predicted by the triangle.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\begin{aligned} (x+y)^0 &= 1 \\ (x+y)^1 &= 1\,x + 1\,y \\ (x+y)^2 &= 1\,x^2 + 2\,xy + 1\,y^2 \\ (x+y)^3 &= \underline{1\;\;}\,x^3 + \underline{3}\,x^2\,y + \underline{3}\,xy^2 + \underline{1}y^3 \\ (x+y)^4 &= \underline{\;\;\;\;}\,x^4 + \underline{\;\;\;\;}\,x^3y + \underline{\;\;\;\;}\,x^2y^2 + \underline{\;\;\;\;}\,xy^3 + y^4 \end{aligned}$$

Check:

$$\begin{aligned} (x+y)^3 &= (x+y)\,(x+y)\,(x+y) \\ &= \\ &= \\ (x+y)^4 &= (x+y)\,(x+y)\,(x+y)\,(x+y) \end{aligned}$$

Following both Pascal's triangle and the pattern of x and y powers, see if you can predict $(x+y)^5$ and $(x+y)^6$.

$(x^5 + 5\,x^4y + 10\,x^3y^2 + 10\,x^2y^3 + 5\,xy^4 + y^5)$

$(x^6 + 6\,x^5y + 15\,x^4y^2 + 20\,x^3y^3 + 15\,x^2y^4 + 6\,xy^5 + y^6)$

REFERENCES: Kenney, Margret. *The Incredible Pascal Triangle.* Chestnut Hill, Mass.: Boston College Press, 1976.

PATTERNS IN REPEATING DECIMALS

GRADE LEVEL/ COURSE: 5–6, 7–8

MATERIALS: Paper, pencil, worksheet on repeating decimals, and calculator (at teacher's discretion).

INTRODUCTION/ OBJECTIVE: This activity is designed:

1. To provide additional experience in changing fractions to decimals.
2. To allow students to observe patterns in repeating decimals.
3. To encourage students to draw conclusions about patterns in repeating decimals.
4. To encourage students to analyze denominators of fractions written in lowest terms that generate repeating decimals. (Denominators of these fractions whose prime factorizations consist exclusively of 2s and 5s do not repeat.)
5. To direct students to observe answers that appear more than once: ½ = .5; 3/6 = .5 or ⅓ = $.\overline{3}$; 4/12 = $.\overline{3}$.

PROCEDURE: This activity is best suited to group instruction followed by independent work.

ACTIVITY:

Step 1. Give students a worksheet drawn from the included list of fractions.

Step 2. Have the students choose a few fractions to convert to decimals that they think will be non-repeating (½ = .5; ¾ = .75).

Step 3. Review briefly how to write repeating decimals (⅔ = $.\overline{6}$; ⅚ = $.8\overline{3}$).

Step 4. Have students manually determine the decimal equivalent to fractions whose numerators are one.

Step 5. Students may use calculators to find decimals equivalent to all other fractions.

Step 6. Have students determine whether the resulting decimal equivalent will be repeating or non-repeating by analyzing the factors of denominators.

Step 7. Encourage students to look for patterns: (1/7 = $.\overline{142857}$ 2/7 = $.\overline{285714}$ same digits and same order beginning with different digits in the order;

1/9 = $.\overline{1}$ 2/9 = $.\overline{2}$ 3/9 = $.\overline{3}$

EXTENDED LEARNING:

1. Fractions with denominators greater than 13 may be used.
2. Conclude that repeating decimals are rational numbers.

PATTERNS OF REPEATING DECIMALS

LIST OF FRACTIONS

¼ = .25
½ = .5
¾ = .75
⅓ = $.\overline{3}$
⅔ = $.\overline{6}$

⅕ = .2
⅖ = .4
⅗ = .6
⅘ = .8

⅙ = $.1\overline{6}$
2/6 = $.\overline{3}$
3/6 = .5
4/6 = $.\overline{6}$
⅚ = $.8\overline{3}$

$\frac{1}{7} = .\overline{142857}$

$\frac{2}{7} = .\overline{285714}$

$\frac{3}{7} = .\overline{428571}$

$\frac{4}{7} = .\overline{571428}$

$\frac{5}{7} = .\overline{714285}$

$\frac{6}{7} = .\overline{857142}$

$\frac{1}{8} = .125$

$\frac{2}{8} = .25$

$\frac{3}{8} = .375$

$\frac{4}{8} = .5$

$\frac{5}{8} = .625$

$\frac{6}{8} = .75$

$\frac{7}{8} = .875$

$\frac{1}{9} = .\overline{1}$

$\frac{2}{9} = .\overline{2}$

$\frac{3}{9} = .\overline{3}$

$\frac{4}{9} = .\overline{4}$

$\frac{5}{9} = .\overline{5}$

$\frac{6}{9} = .\overline{6}$

$\frac{7}{9} = .\overline{7}$

$\frac{8}{9} = .8$

$\frac{1}{10} = .1$

$\frac{2}{10} = .2$

$\frac{3}{10} = .3$

$\frac{4}{10} = .4$

$\frac{5}{10} = .5$

$\frac{6}{10} = .6$

$\frac{7}{10} = .7$

$\frac{8}{10} = .8$

$\frac{9}{10} = .9$

$\frac{1}{11} = .\overline{09}$

$\frac{2}{11} = .\overline{18}$

$\frac{3}{11} = .\overline{27}$

$\frac{4}{11} = .\overline{36}$

$\frac{5}{11} = .\overline{45}$

$\frac{6}{11} = .\overline{54}$

$\frac{7}{11} = .\overline{63}$

$\frac{8}{11} = .\overline{72}$

$\frac{9}{11} = .\overline{81}$

$\frac{10}{11} = .\overline{90}$

$\frac{1}{12} = .08\overline{3}$

$\frac{2}{12} = .1\overline{6}$

$\frac{3}{12} = .25$

$\frac{4}{12} = .\overline{3}$

$\frac{5}{12} = .41\overline{6}$

$\frac{6}{12} = .5$

$\frac{7}{12} = .58\overline{3}$

$\frac{8}{12} = .\overline{6}$

$\frac{9}{12} = .75$

$\frac{10}{12} = .8\overline{3}$

$\frac{11}{12} = .91\overline{6}$

$\frac{1}{13} = .\overline{076923}$

$\frac{2}{13} = .\overline{153846}$

$\frac{3}{13} = .\overline{230769}$

$\frac{4}{13} = .\overline{307692}$

$\frac{5}{13} = .\overline{384615}$

$\frac{6}{13} = .\overline{461538}$

$\frac{7}{13} = .\overline{538461}$

$\frac{8}{13} = .\overline{615384}$

$\frac{9}{13} = .\overline{692307}$

$\frac{10}{13} = .\overline{769230}$

$\frac{11}{13} = .\overline{846153}$

$\frac{12}{13} = .\overline{923076}$

PERFECT NUMBERS

GRADE LEVEL/ COURSE: Algebra I

INTRODUCTION/ OBJECTIVE: This unit may be used for expansion after a unit in number theory that involves work with rational, irrational and prime numbers. Students must be familiar with the elementary properties of natural numbers.

1. Given a certain number, the student will determine if it is a perfect number.
2. Students will extend their understanding of perfect numbers.

PROCEDURE: This should be a classroom activity led by the teacher.

HISTORY: The teacher may begin with a short history of number theory. This should be accomplished in a spontaneous manner to stir the interest of the students. Number theory was studied by a school of ancient Greek philosophers led by Pythagoras about 500 B.C. The Pythagorean school considered the natural numbers to be the key to understanding the universe. They had a solemn reverence for these numbers and not only used them for the basis of their philosophical studies but for their religious rites. They imparted to natural numbers humanistic and mystic properties, considering certain numbers more beautiful and significant than others. They applied to numbers adjectives that are frequently associated with people such as friendly, perfect, natural, rational, and irrational. Even today some people have their lucky and unlucky numbers.

A natural number is defined as *perfect* if it is the sum of its divisors other than itself. For example, 6 and 28 are perfect numbers since $6 = 3 + 2 + 1$, and $28 = 14 + 7 + 4 + 2 + 1$. The ancient Greeks knew about the first four perfect numbers — 6, 28, 496, and 8,128 and endowed them with mystical properties. In answer to why God created the universe in 6 days, or why the circle of the moon takes 28 days, the ancient Greek replied, "because these numbers are perfect!" In classical Greek numerology, 6 was considered the most beautiful of all numbers — 6 represented marriage, health, and beauty, being the sum of its own parts. It also represented the goddess of love, Venus, for it is the product of 2, which represents female, and 3, which represents male. Aurelius Augustinus (St. Augustine, A.D. 354–430) has said,

> "Six is a number perfect in itself, and not because God created all things in six days; rather the converse is true; God created all things in six days because this number is perfect, and it would have been perfect even if the work of the six days did not exist."

After examining the first four perfect numbers and after much philosophizing, the ancients made the following conjectures:

1. There are infinitely many perfect numbers.
2. The perfect numbers end alternately in 6 and 8.
3. There is exactly one perfect number with a given number of digits.
4. Every perfect number is even.

It is believed that these conjectures were more empirical and philosophical, than based on proof, since some of them have turned out to be false. The conjectures as they stand today are:

1. Only 24 perfect have been found and it is not known whether or not there are infinitely many perfect numbers.
2. Although all even perfect numbers end in 6 and 8, they do not end alternately in 6 and 8.
3. This conjecture has been proven false.
4. The 24 known perfect numbers are even, but it is not known whether *all* perfect numbers are even. However, it has been proven that if there are odd perfect numbers, they must be greater than 10^{20}.

ACTIVITY: There is no known formula that gives all the primes; although, there is a formula that gives all the even perfect numbers. An examination of the prime factors of the first four perfect numbers will be useful in developing such a formula.

$$6 = (2)(3) = (2)(2^2 - 1)$$

$$28 = (2^2)(7) = (2^2)(2^3 - 1)$$

$$496 = (2^4)(31) = (2^4)(2^5 - 1)$$

$$8{,}128 = (2^6)(127) = (2^6)(2^7 - 1)$$

After examining the equations above, the student may be led to the conjecture that every even perfect number has the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime. It turns out that not only is this conjecture true, but its converse is also true — i.e., every number of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, is an even perfect number. A proof of this conjecture is relatively easy to follow and may be done at the discretion of the teacher. Such a proof is done by Stanley Gudder in his book, "A Mathematical Journey."

The largest known perfect number, $(2^{19{,}936})(2^{19{,}937} - 1)$, is so large it would take an entire volume to write it out in full. If the size of these numbers increases at the same rate, it would take bookshelves of books to write out in full the next perfect number after this number. "Perfect number" is the right term for these numbers — only the perfect number could be so rare.

The 2,500-year quest for perfect numbers is still continuing. Where will it lead? What will it accomplish? The same questions may be applied to why men scale Mt. Everest. The question of the existence of infinitely many perfect numbers might take another 2,500 years to answer.

The teacher may choose to show why even perfect numbers do end alternately in 6 and 8. Every even perfect number has the form $(2^{p-1})(2^p - 1)$, where $2^p - 1$ is prime. Powers of 2 must end in 0, 1, 2, 4, 6, or 8. But a power of 2 cannot end in 0 since it would then be divisible by 5, and no power of 2 has 5 as a factor. If $2^{p-1} = 1$ then $2^p - 1 = 1$ which is not prime. If 2^{p-1} ends in 2, then $2^p = (2)(2^{p-1})$ must end in 4, and so $2^p - 1$ must end in 3. If 2^{p-1} ends in 4 then $2^p = (2)(2^{p-1})$ must end in 8, and so $2^p - 1$ must end in 7. If 2^{p-1} ends in 6 then $2^p = (2)(2^{p-1})$ must end in 2 and so $2^p - 1$ must end in 1. If 2^{p-1} ends in 8 then $2^p = (2)(2^{p-1})$ must end in 6 and $2^p - 1$ must end in 5. A summary of the last digit of 2^{p-1}, $2^p - 1$, and $(2^{p-1})(2^p - 1)$ is given below. The last digit of $2^p - 1$ cannot be 5 if $2^p - 1$ is prime since otherwise $2^p - 1$ would be divisible by 5 (for example, $2^4 - 1 = 15$, is not prime). Hence the 1st, 4th, 6th, and 8th cases in the table below must be eliminated. It thus follows that numbers of the form $(2^{p-1})(2^p - 1)$, where $2^p - 1$ is prime, end in 6 or 8. A table of values of p used to generate more of the 24 known perfect numbers and their last digits can be developed to illustrate that $(2^{p-1})(2^p - 1)$, where $2^p - 1$ is prime, forms numbers ending in 6 or 8.

Notice that a perfect number $(2^{p-1})(2^p - 1)$ ends in 6 if and only if 4 is a factor of $p - 1$ (except for the case $p = 2$).

Table or Chart

p	2^{p-1}	$2^p - 1$	$(2^{p-1})(2^p - 1)$
1	1	1 (not prime)	
2	2	3	6
3	4	7	28
4	8	15 (not prime)	
5	16	31	496
6	32	63 (not prime)	
7	64	127	8128
8	128	255 (not prime)	

EXTENDED LEARNING: The following problems may be assigned for homework or additional classwork as time permits:

1. Show that 12 is not perfect by adding its divisors less than itself.
2. Show that there are no perfect numbers between 6 and 28.
3. Verify that 496 is perfect by adding its divisors less than itself.
4. A natural number is called *magic* if when we add its digits and then add the digits of the resulting number, etc., we finally obtain 1. For example. 298 is magic.
   (a) Show that the first six perfect numbers except 6 are magic.
   (b) Find 10 magic numbers that are not perfect.
5. A natural number is multiplicatively perfect if it equals to the product of its divisors other than itself.
   (a) Show that 6, 8, 27, 35, 125, and 134 are multiplicatively perfect.
   (b) If $N = p^3$, where p is prime, show that n is multiplicatively perfect.

REFERENCES:

Gudder, Stanley. *A Mathematical Journey.* New York: McGraw Hill Book Company, 1976.

POMPEIU'S MAGIC SEVEN

GRADE LEVEL/ COURSE: Algebra I–II

INTRODUCTION/ OBJECTIVE:

1. Students will demonstrate a property of the number seven which leads to a permutation of the digits 0 through 9 such that there exists a one-to-one correspondence between the multiplier digits and the products digits.
2. Students will identify other digits which have this same property.
3. Students will apply this property to division examples where the first few digits on the extreme left of the dividend are unknown.

HISTORY: In 1951, after a lifetime of teaching at the University of Bucharest, Rumanian mathematician Dmitri Pompeiu posed the following puzzle:

> ABC,398,246 is a number in the hundreds of millions whose first three digits, A, B, C, are unknown. This number is exactly divisible by 317. What are the digits A, B, and C?

PROCEDURE: This should be treated as an individual classroom activity which can be completed at home if time is limited. If completed at home, students will need printed instructions.

ACTIVITY: In attempting to solve Pompeiu's Magic Seven problem it should first be elicited from the students that in division, we normally work first with the digits of the dividend that are to the extreme left — just the ones we *don't* know in the puzzle.

Where then do we start?

Have students perform the multiplication of numerals 0 through 9 by seven.

According to Pompeiu, the secret of success depends on two approaches to the problem:

1. Change the division into a multiplication problem so we can deal first with the digits to the far right, which we know.
2. Use the following curious fact about the number seven (which is in the units digit of the number 317):

Seven times table from 0 to 9

$$\begin{array}{rrrrrrrrrr} 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ \times\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & 7 & 14 & 21 & 28 & 35 & 42 & 49 & 56 & 63 \end{array}$$

If we forget the tens digits of these products and look only at the units digits we get the numbers 0, 7, 4, 1, 8, 5, 2, 9, 6, 3 respectively.

Elicit that these digits are all the digits from 0 to 9, but they are switched around into a different sequence — permutation of the digits.

A one-to-one correspondence can now be set up between the multiplier digits and the product digits.

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3 \end{array}$$

Now, returning to our puzzle: 317 times some unknown number equals ABC,398,246. Let's call the unknown number . . . RSTUVWXYZ. We will soon see how many digits are required.

Have students write this out as a multiplication example:

$$\begin{array}{r} \ldots\text{RSTUVWXYZ} \\ \times \qquad 317 \\ \hline \\ \hline \text{ABC,398,246} \end{array}$$

Lead students to discover that seven times Z must be a number that ends in 6. Consulting our one-to-one correspondence, we find that the number Z must be 8 — there is no other possibility. When we replace the Z by 8, we can complete a part of the multiplication.

$$\begin{array}{r} \ldots\text{RSTUVWXY8} \\ \times \qquad 317 \\ \hline 2536 \\ \ldots\text{HJKL} \\ \\ \hline ..\text{ABC398246} \end{array}$$

We now bring in a second row of unknown digits, written as . . . HJKL.

Ask the students, "What can be said about them?"
It is clear that L + 3 = 4. Therefore, L must be 1 and 7 times Y be a number that ends in 1.

Consulting our correspondence, we find Y must be 3. Again there is no other choice. After we write 3 for Y, we can complete the second line of the multiplication.

$$\begin{array}{r} \ldots\text{RSTUVWX38} \\ \times \qquad 317 \\ \hline \\ 2536 \\ 951 \\ \ldots\text{DEFG} \\ \ldots\phantom{\text{ABC398246}} \\ \hline \text{ABC398246} \end{array}$$

Looking at our third row of digits, . . . DEFG, 5 + 5 + G must be a number that ends in 2 and X must be 6.

$$\begin{array}{r} \ldots\text{RSTUVW638} \\ \times \qquad 317 \\ \hline 2536 \\ 951 \\ 1902 \\ \ldots\text{MNOP} \\ \ldots\phantom{\text{ABC398246}} \\ \hline \text{ABC398246} \end{array}$$

Looking at our fourth row of digits, . . . MNOP, 1 (the remainder carried over from the previous column) + 2 + 9 + P must end in 8. Therefore P must be 6 and 7 times W must be a number which ends in 6, namely W = 8.

Have students follow this path through to the end. They should get A = 3, B = 1, C = 3.

Next, have them try to find A and B if AB,528,237 is a number that is exactly divisible by 117. (A = 1, B = 8)

- - - - -

Pompeiu's solution depends crucially upon the permuting power of the multiplication by 7. Do all digits have this power?

— If we were to use 6 as a multiplicand:

$$\begin{array}{rrrrrrrrrr} 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ \underline{\times\ 0} & \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} & \underline{6} & \underline{7} & \underline{8} & \underline{9} \\ 0 & 6 & 12 & 18 & 24 & 30 & 36 & 42 & 48 & 54 \end{array}$$

When the tens digit in the product is ignored, we obtain the correspondence:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 6 & 2 & 8 & 4 & 0 & 6 & 2 & 8 & 4 \end{array}$$

In this case, the second line is not a permutation of the first line, because the odd numbers are not present.

We now know that not all digits have this power, but are there any other digits, in addition to 7, that do have this property?

By experimenting, students should find that 3 and 9 also have this property.

Multiplication by 3 brings about the correspondence:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{array}$$

and multiplication by 9 brings about the correspondence:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

Students should notice how this permutation just switches around the positive integers.

There is one multiplier that is so simple it might be easily overlooked — the number 1, multiplication by which brings about the permutation:

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

This is frequently called the "identity" permutation.

We can now solve a Pompeiu puzzle with a multiplier that ends in a 1, 3 or 9. Have students try the following puzzle:

The number A,BC8,682 is exactly divisible by 213. Find the digits A, B, and C.

Why does this property work for 1, 3, 7 and 9 and not for the other digits?

1, 3, 7 and 9 are the digits that are relatively prime to 10 — that is, they have no factors other than 1 that is in common with 10.

Remind students that we ignored the tens digits in the products when we set up our permutations. In other words, we subtracted from the products all multiples of 10 that were less than the product. This process of subtracting multiples of a fixed number is called "residue arithmetic."

The following is an example of "residue arithmetic":

> If 9 is our fixed number instead of 10, subtracting multiples 9 from 11, 18 and 58 would leave —
> 2 $(11 = 9 \times 1 + 2)$
> 0 $(18 = 9 \times 2 + 0)$
> 4 $(58 = 9 \times 6 + 4)$

The property that applies for the digits 1, 3, 7 and 9 is a special instance of a more general phenomenon: "If we agree to subtract multiples of a fixed number N, then multiplication of the digits by any fixed number relatively prime to N will permute the digits." This theorem can be further investigated by more advanced students.

To illustrate the theorem with an example:

> Suppose we perform residue arithmetic with respect to 9. Select 5 (a number relatively prime to 9) as a multiplicand.

$$\begin{array}{rrrrrrrrr} 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ \times\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 \end{array}$$

Now, subtracting multiples of 9:

0 5 1 6 2 7 3 8 4

This leads to the permutation:

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 5 & 1 & 6 & 2 & 7 & 3 & 8 & 4 \end{array}$$

EXTENDED LEARNING:

1. The number A,B25,815 is exactly divisible by 189. Find A and B.
2. Solve the following "residue arithmetic" problem with respect to "8"
   - **a.** Select a number that is relatively prime to 8.
   - **b.** Use the prime you have just selected as your multiplicand.
   - **c.** Take out the multiples of 8.
   - **d.** Show the permutation this leads to.

REFERENCES:

Davis, Philip J. and Chinn, William G. *3.1416 and All That*. New York: Simon and Schuster. 1969.

PROBABILITY AND PASCAL'S TRIANGLE

GRADE LEVEL/ COURSE: 7–8

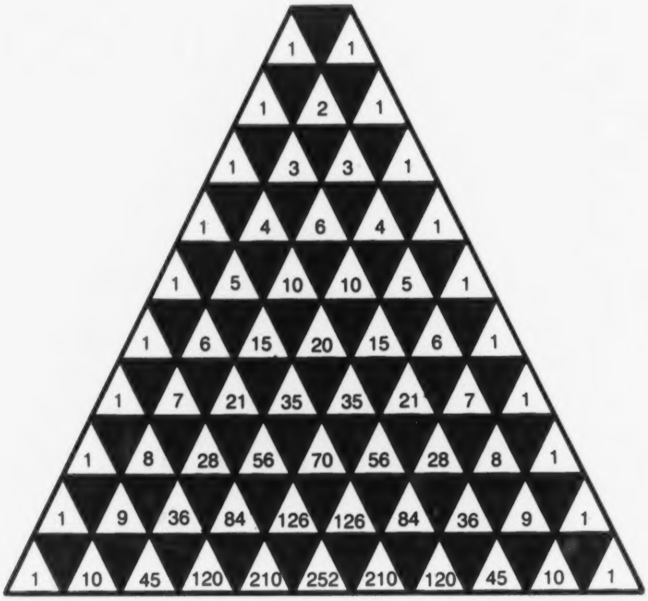
INTRODUCTION/ OBJECTIVE:

1. The student will analyze Pascal's triangle and discover number patterns.
2. The student will be able to determine binomial probability using Pascal's triangle.
3. The student will discover that the coefficients of $(a+b)^n$ are the rows of Pascal's triangle.

HISTORY: Blaise Pascal (1623–1662) is noted for the development of probability theory. He used a form of the triangle below to calculate binomial probability and wrote a book about its properties. Because of his book we think of the triangle as Pascal's; however, various forms of the triangle were used as early as the 12th century in China and Persia.

PROCEDURE: These activities should involve interaction between the teacher and the entire class. The activities can be used separately and integrated with the regular curriculum.

ACTIVITY 1: Provide copies of this triangle of numbers and ask students what patterns they notice. Allow discussion of the patterns they see. If necessary, guide students into the discovery that the sum of each row is a power of 2, e.g., the sum of the eighth row is two to the eighth power, which is 256.



ACTIVITY 2: Have students toss coins and construct a table of all possible results for a varying number of coins. (Illustration below)

NUMBER OF HEADS

NO. OF COINS	0	1	2	3	4	5
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	

If necessary, lead students to realize that this is a form of Pascal's triangle.

ACTIVITY 3: Define successful probability as $\frac{\text{the number of successful outcomes.}}{\text{the number of possible outcomes}}$

Illustration: Out of 3 tosses of a coin, what is the probability that exactly 2 out of the 3 will be heads? The possible outcomes are:

H H H	H H T	T T H	T T T
	H T H	T H T	
	T H H	H T T	

The successful outcomes are H H T, H T H and T H H. The probability is 3/8.

ACTIVITY 4:

Step 1. Lead students into an explanation of approximate binomial probability where each sample event (e.g. H) is one of two nearly equally likely outcomes. Examples: flipping of a coin, sex of a baby, guess on a true-false test.

Illustration: Display a photo (from *Mathematics,* Time-Life chapter on probability) showing a family with a large number of children of the same sex. Emphasize that the probability that a child is a girl is approximately 50%. Note the analogy between girl vs. boy, and the heads-tail model provided above. Ask students to find the probability that a 3-child family will contain 2 girls. (Answer 3/8) Using the photo, ask students to estimate the probability of the combination pictured.

Step 2. Guide students to solve a simpler problem, discovering the numerator and denominator in Pascal's triangle.

Illustration:	**Sum of Row**
1 1	$2 = 2^1$
1 2 1	$4 = 2^2$
1 3 ③ 1	⑧ $= 2^3$

Probability of 2 girls in 3 children = 3/8.
(On row 3, count "0, 1, 2")

Step 3. Have students solve the problem in the photo. (Probability of 5 girls in 10 children is 252/1024). Allow student discussion of interesting girl-boy family combinations in their experiences. Have them determine the probability of the more intriguing combination, using Pascal's triangle.

EXTENDED LEARNING:

A. Have students simplify $(b+g)^2$ and record the coefficients. Do the same for $(b+g)^3$ and $(b+g)^4$, etc., guiding the discovery that the coefficients form the rows of Pascal's triangle.

B. Explore the fact that the rows of Pascal's triangle reveal the number of combinations of n things taken r at a time. For example, when $n=4$ and $r=2$ we may consider the 2-letter combinations of the 4 letters URYZ:

UR	RY	YZ
UY	RZ	
UZ		

There are 6 such combinations. The number 6 is the third entry in the 4th row of Pascal's triangle. The 4 letters (URYZ) imply the 4th row. From the fact that there are 2 letters in each combination, we infer that we should count along the row, "0 elements, 1 element, $\underline{2}$ elements, as shown below.

1 1

URYZ 1 2 1

(row 4) 1 3 3 1

1 4 ⑥ 4 1

UR ↑ ↑ ↑

Count: 0, 1, 2

There are 6 combinations of 4 letters taken 2 at a time.

Ask the students, "If you have 2 free tickets to give to 2 of your 3 best friends, how many combinations of 2 of these 3 friends could you choose?". (Answer: The number combinations of 3 things taken 2 at a time is the third number of the the 3rd row, i.e., 3.)

C. Explore the fact that Activity 4 is based on the formula $P=C(n,r)\ p^r q^{n-r}$ where

C(n,r) is the number of combinations of n things taken r at a time (producing the numerator 3, because $n=3$, $r=2$)

$p^r q^{n-r}$ produces the denominator 8

p is the probability that a child is a girl
q is the probability that a child is not a girl

So $p^r q^{n-r}$ is $(1/2)^2\ (1/2)^1 = 1/8$

Thus $P = 3(1/8) = 3/8$

REFERENCES:

Kenney, Margaret. *The Incredible Pascal Triangle.* Chestnut Hill, Mass.: Boston College Press, 1976.

Bergamini, David. *Mathematics.* New York: Time-Life Books, 1970.

PROBLEM SOLVING STRATEGIES

GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVE: Students will have an understanding of some problem solving procedures. This activity provides students with a strategy for solving problems through simple deductive logic. Also, it builds on their ability to think systematically and abstractly.

PROCEDURE: Fourth and fifth graders are often being introduced to problem solving techniques (that involve more than simple arithmetic) for the first time. It will help the students to be guided through several problems before they are assigned problems to solve independently.

ACTIVITY:

Step 1. Write the following problem on the board.

Bob's father is three times older than Bob, but in twelve years Bob's father will only be twice as old as Bob. How old is Bob now?

Step 2. Make a chart with four columns and label:

Bob's Age	Bob's Father's Age	Bob's Age in 12 years	Bob's Father's Age in 12 years

Step 3. Have a student guess an age for Bob. Student guesses 10. Fill in and complete the first line of the chart.

Bob's Age	Bob's Father's Age	Bob's Age in 12 years	Bob's Father's Age in 12 years
10	30	22	42

Step 4. Explain to students how you got these numbers.

Step 5. Since 42 is not twice 22, we can now conclude that Bob is not presently 10 years old. Ask for another student to guess an age.

Bob's Age	Bob's Father's Age	Bob's Age in 12 years	Bob's Father's Age in 12 years
10	30	22	42
20	60	32	72

Step 6. We can now conclude that Bob's age is not 20. But it is between 10 and 20. The students will continue to fill in possible ages until they find the right age.

Step 7. The finished product could look similar to this:

Bob's Age	Bob's Father's Age	Bob's Age in 12 years	Bob's Father's Age in 12 years
10	30	22	42
20	60	32	72
11	33	23	45
12	36	24	48

EXTENDED LEARNING:

There are many publications with a variety of problem solving procedures. *Mind Benders* by Anita Harnadek contains problems that are challenging to highly capable students.

PYTHAGOREAN THEOREM

GRADE LEVEL/ COURSE: 7, 8, Geometry

MATERIALS: Hand-outs and scissors.

INTRODUCTION/ OBJECTIVE: Students will develop an understanding of the Pythagorean Theorem. The student will derive the formula $c^2 = a^2 + b^2$ by using several cut up examples. This activity provides students with an opportunity to develop their own concept of the Pythagorean Theorem.

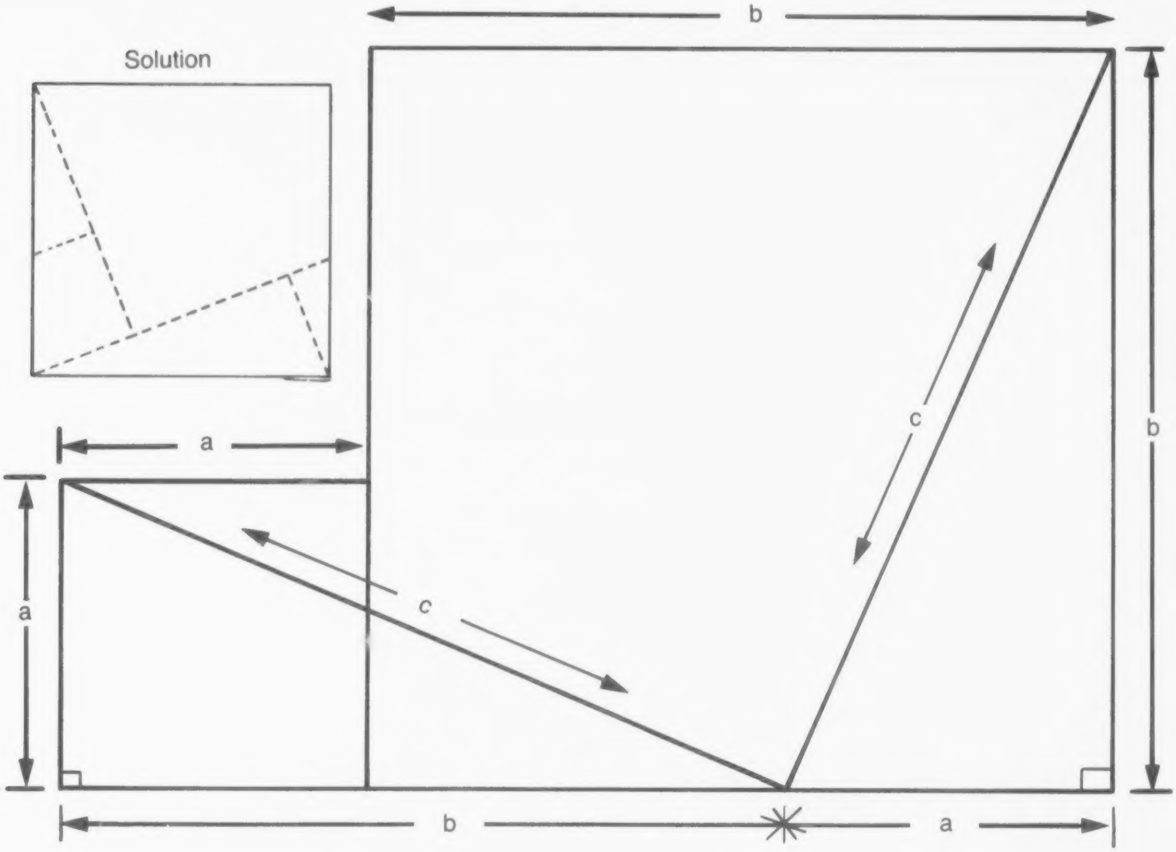
PROCEDURE: This activity can be done with the whole class or a small group. If done with the whole class, students may work in pairs.

ACTIVITY 1:

Step 1. Cut out two squares with sides a and b.

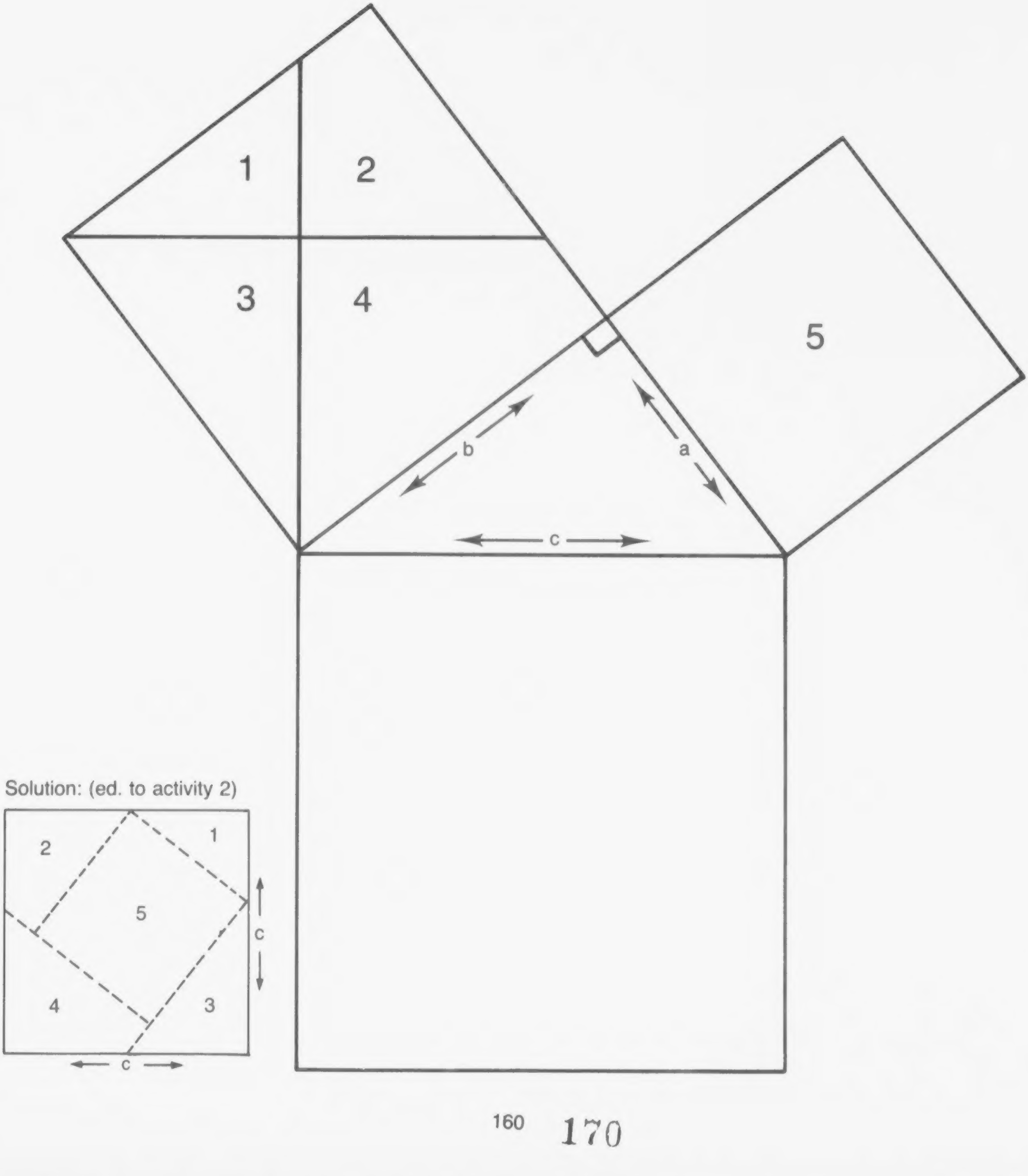
Step 2. Arrange the squares as shown below and cut along the lines marked "C".

Step 3. Rearrange the pieces to show $c^2 = a^2 + b^2$.

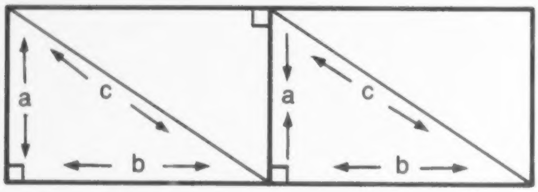


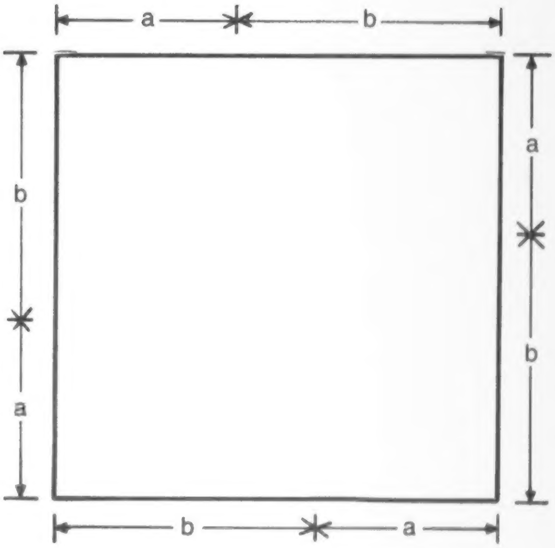
ACTIVITY 2: **Step 1.** Cut out pieces 1, 2, 3, 4, and 5.

Step 2. Put them together to show $c^2 = a^2 + b^2$.



Solution: (ed. to activity 2)





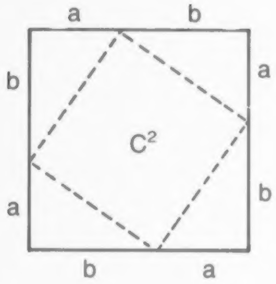
ACTIVITY 3:

Step 1. Cut out the four triangles in the rectangle. Notice that the area of the rectangle is $a(2b) = 2ab$.

Step 2. By leaving c^2 uncovered, place the triangles in the large square to show:

$$\begin{aligned}(a+b)^2 &= c^2 + 2ab\\ a^2 + 2ab + b^2 &= c^2 + 2ab\\ a^2 + b^2 &= c^2\end{aligned}$$

Solution:



REULEAUX TRIANGLES AND OTHER CURVES OF CONSTANT WIDTH

GRADE LEVEL/ COURSE: Geometry

MATERIALS: Compass, straightedge and paper.

INTRODUCTION/ OBJECTIVE: Students will identify and construct the Reuleaux triangle and a pentagonal curve of constant width using a compass and a straight edge. The equilateral triangle and the regular pentagon used to form the curves may be provided for the student if they have not learned to construct them.

A closed curve of constant width is a set of points in a plane that enclose a region that when rotated between 2 parallel surfaces will always maintain contact with both surfaces. The width of the curve is the measure of its diameter, the segment from edge to edge passing through the center of the polygon used to generate the curve. The simplest of these curves is the circle. Other curves of constant width can be constructed from any regular polygon with an odd number of vertices by connecting consecutive vertices with arcs centered at the vertex opposite the side joining the two vertices. The curve constructed from an equilateral triangle is called a Reuleaux triangle after the German scientist Franz Reuleaux.

PROCEDURE: The student will do the following activities in the class.

ACTIVITY 1: **Construct a Reuleaux triangle.**

Step 1. Construct an equilateral triangle. Label its vertices A, B and C.

Step 2. With a compass using point A as a center and AB as a radius, construct an arc from B to C.

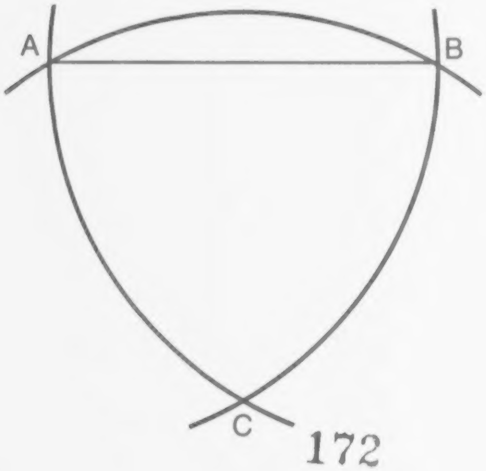
Step 3. Using point B as a center and AB as a radius construct an arc from A to C.

Step 4. Using point C as a center and AB as a radius construct an arc from A to B.

Step 5. The resulting figure is a Reuleaux triangle.

SOLUTION:

REULEAUX TRIANGLE



ACTIVITY 2: **Construct a pentagonal curve of constant width.**

Step 1. Construct a regular pentagon. Label its vertices A, B, C, D, E.

Step 2. Using point A as a center and $\overline{AC}$ as a radius construct an arc from C to D.

Step 3. Using point B as a center and $\overline{BD}$ as a radius construct an arc from D to E.

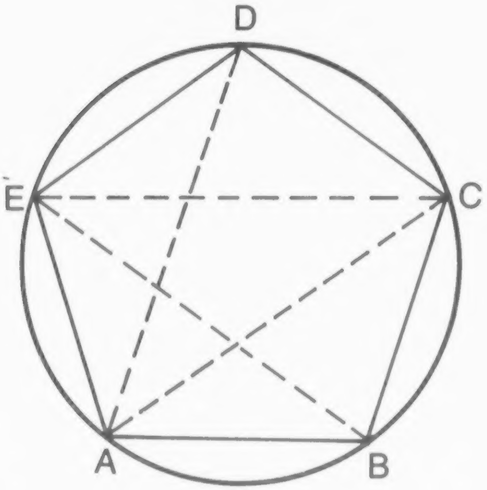
Step 4. Using point C as a center and $\overline{CE}$ as a radius construct an arc from E to A.

Step 5. Using D for a center and $\overline{AD}$ as a radius construct an arc from A to B.

Step 6. Using E as a center and BE as a radius construct an arc from B to C.

Step 7. The resulting figure is a pentagonal curve of constant width. **Note:** The radii for all of the arcs are congruent.

SOLUTION: **PENTAGONAL CURVE**



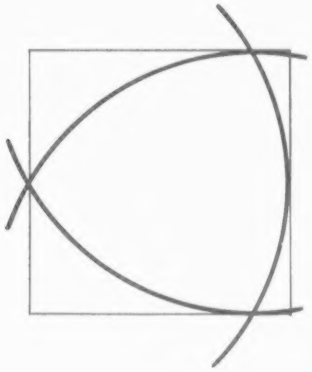
EXTENDED LEARNING:

If the Reuleaux triangle is placed in the interior of a square with sides congruent to the side of the triangle from which the curve is constructed, the curve will always maintain contact with all four sides of the square simultaneously. This characteristic of the Reuleaux triangle was used in 1914 to manufacture a drill bit that would bore a square hole.

During the 1970's the Wankel rotary engine was introduced on the automobile market. The rotor for this engine was designed from the Reuleaux triangle because it allowed constant contact with the walls of the engine and yet allowed space for the fuel and air to be injected.

Students could be asked to find the area of a Reuleaux triangle associated with an equilateral triangle with side of length x.

REULEAUX TRIANGLE IN A SQUARE



SEQUENCES AND SERIES

GRADE LEVEL/ COURSE: Algebra II

INTRODUCTION/ OBJECTIVE: The students should find sums of arithmetic series from the n^{th} term of the partial sum of the given series and establish a relationship between the sum of the arithmetic series and the sequence of partial sums of the series.

PROCEDURE: The student is asked to generate a sequence of partial sums of an arithmetic series and then attempt to discover a general expression for the n^{th} term of the sequence. This n^{th} term will then be the sum of the first 'n' terms of the series.

Sometimes it is rather difficult to determine the general expression in the first sequence of partial sums so the student is asked to generate a pair of series which together equals the original. The first series should be generated by exhibiting the sum of the 1st, 3rd, 5th, 7th, . . ., 2n–1 terms. The second series should be generated by exhibiting the sum of the 2nd, 4th, 6th, . . ., 2n terms.

Next, the student is asked to generate sequences of partial sums for both partial series and attempt to find a general expression for each of the sequences.

Finally, the student is asked to determine the sum of the 1st n terms of the original series by examining the sum of the 1st, n terms of each of the partial series.

ACTIVITY 1: To find the sum of the 1st n positive integers —

Consider the following series:

$1 + 2 + 3 + 4 + 5 + \ldots + n + \ldots$

Generate a sequence of partial sums.

$1, 3, 6, 10, 15, 21, 28, \ldots$

The n^{th} term of this sequence will be the sum of the 1st n terms of the positive integers.

Generate a pair of partial series which together equals the original.

(a) $1 + 3 + 5 + 7 + 9 + \ldots + 2n - 1 + \ldots$

(b) $2 + 4 + 6 + 8 + 10 + \ldots + 2n + \ldots$

Generate a sequence of partial sums for each of the partial series and then determine the n^{th} term for each sequence.

(a) $1, 4, 9, 16, 25, \ldots n^2, \ldots$

(b) $2, 6, 12, 20, 30, 42, \ldots, ?, \ldots$

Observe that the terms of sequence (b) are greater than the terms of the sequence exhibited in (a).

Hence, the n^{th} term of (b) must be greater than the n^{th} term of a. Sequence (b) can be written in the form.

$1^2 + 1, 2^2 + 2, 3^2 + 3, 4^2 + 4, 5^2 + 5, 6^2 + 6, \ldots, n^2 + n$

Observe that the n^{th} term of the sequence is $n^2 + n$ or $n(n + 1)$.

Therefore, each successive term of the sequence 2, 6, 12, 20, 30, 42, . . ., n(n + 1), . . . is respectively twice each successive term of the first sequence of partial sums. 1, 3, 6, 10, 15, 21, . . . ?, . . .

Hence, the n^{th} term of this sequence is $\frac{n(n+1)}{2}$

$$1 + 2 + 3 + 4 + 5 + \ldots + n = \frac{n(n+1)}{2}$$

Thus, we have determined the sum of the 1^{st} n terms of the original series.

Next, let's consider sums of other arithmetic series in a similar manner.

ACTIVITY 2: To find the sum of the 1^{25} n positive multiples of three.

Consider the series:
$3 + 6 + 9 + 12 + 15 + \ldots + 3n + \ldots$

Generate a sequence of partial sums.
3, 9, 18, 30, 45, . . .

Attempt to find a general expression in n for the n^{th} term of the sequence. If the n^{th} term cannot readily be determined, generate a pair of partial series which together equals the original.

(a) $3 + 9 + 15 + 21 + 27 + \ldots + 3(2n - 1) + \ldots$

(b) $6 + 12 + 18 + 24 + 30 + \ldots + 6n + \ldots$

Recall that ½ of the n^{th} term of the sequence of partial sums for the 2^{nd} series produces the sum of the first n terms of the original series.

Hence, we shall generate that sequence only.

6, 18, 36, 60, 90, . . .

6•1, 6•3, 6•6, 6•10, 6•15, . . .

Since the n^{th} term for the sequence 1, 3, 6, 10, 15, . . . is $\frac{n(n+1)}{2}$, the n^{th} term for the sequence in question is

$$\frac{6n(n+1)}{2} = 3n(n+1).$$

Therefore, $3 + 6 + 9 + 12 + 15 + \ldots + 3n = \frac{3n(n+1)}{2}$

EXTENDED LEARNING:

1. To find the sum of the first n positive multiples of seven,

   Consider the series:

   $7 + 14 + 21 + 28 + 35 + \ldots + 7n + \ldots$

   Consider the 2^{nd} partial series:
   $14 + 28 + 42 + 56 + \ldots + 14n + \ldots$

   Generate a sequence of partial sums for the 2^{nd} partial series and find the n^{th} term of that sequence.

   14, 42, 84, 140, . . .

   14•1, 14•3, 14•6, 14•10, . . .

   Hence, the n^{th} term for the sequence is $\frac{14n(n+1)}{2} = 7n(n+1)$

   Therefore, $7 + 14 + 21 + 28 + \ldots + 7n = \frac{7n(n+1)}{2}$

2. To find the sum of the 1st n terms of a general arithmetic series,

Consider the following series which was arbitrarily chosen:
$8 + 17 + 26 + 35 + 44 + \ldots + (9n - 1) + \ldots$

Generate a sequence of partial sums.

8, 25, 51, 86, 130, . . .

Consider the 2nd partial series.

$17 + 35 + 53 + 71 + 89 + \ldots 18n - 1 + \ldots$

Generate a sequence of partial sums and then find the nth term of the sequence.

17, 52, 105, 176, 265, . . .

$17 \cdot 1 + 0,\ 17 \cdot 3 + 1,\ 17 \cdot 6 + 3,\ 17 \cdot 10 + 6,\ 17 \cdot 15 + 10, \ldots$

Observe that the sequence 1, 3, 6, 10, 15, . . ., $\frac{n(n+1)}{2}$ always occurs.

Observe also that here the sequence occurs twice, i.e., as a factor of 17 in each successive term and as an addend in each successive term beginning with the second term.

Hence, the general term of the sequence is

$$\frac{17n(n+1)}{2} + \frac{(n-1)[(n-1)+1]}{2}$$

$$= \frac{17n(n+1) + (n-1)n}{2}$$

$$= \frac{17n^2 + 17n + n^2 - n}{2}$$

$$= \frac{18n^2 + 16n}{2}$$

$$= \frac{2n(9n+8)}{2}$$

$$= n(9n+8)$$

Since the sequence of partial sums is not exactly twice the original sequence of partial sums we must take a closer look.

Original sequence of partial sums:

8, 25, 51, 86, 130, . . .

Sequence of partial sums of the 2nd partial series:

17, 52, 105, 176, 265, . . .

$$\frac{17-1}{2}, \frac{52-2}{2}, \frac{105-3}{2}, \frac{176-4}{2}, \frac{265-5}{2}, \ldots$$

Therefore, $8 + 17 + 26 + 35 + 44 + \ldots + (9n - 1)$

$$= \frac{n(9n + 8) - n}{2}$$

$$= \frac{n(9n + 8 - 1)}{2}$$

$$= \frac{n(9n + 7)}{2}$$

3. Conclusion

   It appears that the sequence of partial sums of the original series, 1, 3, 6, 10, 15, 21, . . ., $\frac{n(n + 1)}{2}$, . . .

   shows up either as a factor, or as a factor and an addend in the general expression representing the sums of the 1st n terms of the series $1 + 2 + 3 + 4 \ldots$

   The students may be asked to try this procedure on several other arithmetic series to investigate this conjecture.

   The students may choose to use this procedure on the general arithmetic series:

   $a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (a + nd) + \ldots$

SEQUENCE USING TABLES OF DIFFERENCE

GRADE LEVEL/ COURSE: 7, 8

INTRODUCTION/ OBJECTIVE: This activity is designed for students who are familiar with the concept of sequence.

The purpose of this activity is to generate and complete sequences by using tables of differences.

PROCEDURE: The teacher will work with the entire class to illustrate how a table of differences may be used to determine the next number in a given sequence, using several examples. Students should then be allowed to try the process individually. Next, through examples, the teacher should show students how to generate their own table of differences. Finally, have the students share their own generated sequences with the class.

ACTIVITY 1: Give the students a sequence of numbers and illustrate how a table of differences can be used to find the next number in the sequence. For example: 2, 4, 8, 16, 31, 57, __.

Step 1: Find the difference between successive numbers until a constant is determined.

2, 4, 8, 16, 31, 57, __
2 4 8 15 26
2 4 7 11
2 3 4
1 1

(Notice that the difference between 2 and 4 is 2, between 4 and 8 is 4, between 8 and 16 is 8, between 16 and 31 is 15 producing the second row)

Step 2: Using the constant, the bottom number, 1, calculate by addition the next number in each row, moving from the bottom row of the table to the top.

2, 4, 8, 16, 31, 57, 99
2 4 8 15 26 42
2 4 7 11 16
2 3 4 5
1 1 1

Step 3: Some examples for students would be:

(1) 2, 5, 15, 36, 73, 132, __
(220 is the next number.)

(2) 4, 12, 25, 45, 74, __
(114 is the next number.)

(3) 3, 10, 19, 36, 72, 143, __
(270 is the next number.)

ACTIVITY 2: Have students generate their own table of differences, and the resulting sequence.

Step 1: Choose a constant for the bottom row of your table of differences. (In the example below, 3 is used.)

Step 2: Choose a beginning number for the second row. (In the example below, 4 is used.) Complete this step by adding the constant from the bottom row.

Step 3: Again the first number must be chosen and the row completed by adding the numbers from the row below. (In the example below, 9 was chosen as the beginning number.)

Step 4: This procedure can be repeated any number of times, producing a new sequence in each row.

Example:

5 14 27 47 77
9 13 20 30
4 7 10
3 3

Step 5: The students could be asked to exchange their final sequence (without the table of differences) and have another student produce the next number in the sequence.

REFERENCES:

Posamentier, Alfred S. and Stepelman, Jay. *Teaching Secondary School Mathematics: Techniques and Enrichment Units.* Columbus, Ohio: Merrill Publishing Co., 1981.

SHORT CUT ALGORITHM FOR MULTIPLES OF FIVE

GRADE LEVEL/ COURSE: 5–6

INTRODUCTION/ OBJECTIVE: Students will be able to multiply and divide by 25, 50, 125, and 250 using a short cut method.

It is expected that students will know how to divide by 10, 100, 1000, etc. by moving the decimal point as many places to the left as there are zeros in the divisor. It is also expected that they will know that to multiply by 10, 100, 1000, etc. they must attach as many zeros to the multiplicand as there are in the multiplier. If the multiplicand is a decimal, they should move the decimal point as many places to the right as there are zeros in the multiplier.

PROCEDURE: Ask the students to solve this problem using conventional division. If there is a remainder, it should be expressed as a decimal.

$248 \div 25$ $113 \div 25$

ACTIVITY: After the students have solved this using the conventional method, they will probably welcome an easier course. Show that to divide by 25, multiply the integer by 4 and then divide the product by 100.

$248 \div 25$
$248 \times 4 = 992$
$992 \div 100 = 9.92$

$113 \div 25$
$113 \times 4 = 452$
$452 \div 100 = 4.52$

To teach the short cut to division by 50, ask the students to solve this, using conventional division.

$845 \div 50$ $648 \div 50$

After the students have solved these, expressing the remainders as decimals, show the short cut. To divide by 50, multiply by 2 and divide by 100.

$845 \div 50$
$845 \times 2 = 1690$
$1690 \div 100 = 16.9$

$648 \div 50$
$648 \times 2 = 1296$
$1296 \div 100 = 12.96$

Using the same procedures, show division by 125.

$243 \div 125$ To divide by 125, multiply by 8 and divide by 1000.

$243 \times 8 = 1944$
$1944 \div 1000 = 1.944$

Conversely, the same methods can be applied to multiplication by 25, 50, 125, 250.

Ask the students to solve the following equations using the conventional method.

284×25 4965×25

After the students have solved the problems, show them the short cut.

284×25	To multiply a number by 25 divide the number by 4, expressing the remainder as a decimal. Then multiply the quotient by 100. $284 \div 4 = 71$ $71 \times 100 = 700$
4965×25	$4965 \div 4 = 1241.25$ $1241.25 \times 100 = 124125$

Continuing with the same steps, have the students multiply 87645×50.

When they have done, show them the short cut. To multiply a number by 50, divide the number by 2 and then multiply by 100.

87654×50	$87654 \div 2 = 43827$ $43827 \times 100 = 4382700$
973×50	$973 \div 2 = 486.5$ $486.5 \times 100 = 48650$

Using the converse method of division by 125, show the students that to multiply by 125 you divide by 8 and then multiply by 1000.

896×125	$896 \div 8 = 112$ $112 \times 1000 = 112000$

After practicing division and multiplication by the short cut method, the students should be able to solve the following on their own.

2875×250 $842 \div 250$

If the students appear to be experiencing difficulty, they can be helped to see the relationship between the numbers used in the short cut methods. You can show that division by 25 involved the number 4 and 100 and that 100 divided by 4 is 25. Show that the multiplication by 125 involved the numbers 8 and 1000 and division of 1000 by 8 results in 125. Hopefully, the students will see that the numbers they need are 8 and 1000. Once they arrive at the correct numbers, they just follow the formats which have already been established.

REFERENCES:

Laderman, Jack. *Short Cuts in Arithmetic*. New York: Jayel Publishing Company, 1936.

SHORT CUTS IN ARITHMETIC

GRADE LEVEL/ COURSE: 7–8

INTRODUCTION/ OBJECTIVE: Given specific multiplication and addition problems to compute, students will use the techniques described.

PROCEDURE: Each of the following activities may be used as a separate assignment.

ACTIVITY 1: **The Cashier's Method of Adding:**

In adding a series of large numbers write the sums of each column one under the other, each one space to the left of that above. This will eliminate carrying over of numbers and will make the work far easier to check.

8496	15	sum of the ones place
7241	23	sum of the tens place
8312	13	sum of the hundreds place
3496	26	sum of the thousands place
27,545	27,545	

ACTIVITY 2: **The Cross Multiplication Rule**

To multiply any two-digit number by another two-digit number first multiply the two units digits. Write the units digit of the product as the units digit of the answer. Then cross multiply and add the products plus the number carried over from the first multiplication. Write the units digit of this answer as the tens digit of the final answer. Multiply the tens digits of the original numbers and add what was carried over from the second addition. This forms the entire answer.

27	$2 \times 7 = 14$	write the 4 and carry the 1
× 32	$1 + (2 \times 2) + (3 \times 7) = 26$	write the 6 and carry the 2
864	$2 + (3 \times 2) = 8$	write the 8

ACTIVITY 3: **The Excess Rule**

To multiply two numbers each slightly larger than 100, add the excess over 100 of either number to the entire other number, then suffix the product of the two excesses. If the product is a one-digit number, place a zero in front of it.

113	The excesses are 13 and 4
	$113 + 4 = 117$
× 104	$13 \times 4 = 52$
11752	

ACTIVITY 4: **The Complement Rule**

To multiply two numbers each a little smaller than 100, subtract the complement of one number from the other number and then suffix the product of the two complements. If the product of the two complements is a one-digit number, place a zero in front of it. The complement of a number is 100 minus the number.

98	100 − 98 = 2
	100 − 93 = 7
× 93	98 − 7 = 91
9114	2 × 7 = 14

ACTIVITY 5: **The Lock Rule**

To multiply a two digit number by 11, add the two digits and place this sum between the two original digits. If the sum of the two digits exceeds 9, put the units digit of this sum between the two numbers, and add 1 to the hundreds digit.

23 × 11 = 253   2 + 3 = 5 which is placed between 2 and 3

67 × 11 = 737   6 + 7 = 13 Place the 3 between the 6 and 7 and add the one to the hundreds digit (the 6)

ACTIVITY 6: **The Ones Rule**

To find the sum of two fractions with ones for numerators, add the denominators to get the numerator of the answer and multiply the denominators to get the denominator of the answer.

$\frac{1}{5} + \frac{1}{6} = \frac{11}{30}$ 5 + 6 = 11
5 × 6 = 30

EXTENDED LEARNING:

1. Have students use the methods taught to compute similar problems.
2. Have students investigate as to which numbers limit the Excess Rule.

REFERENCES:

Franco, Anna. "Taxi-cab Geometry". *Math Survey.* New York: Stuyvesant High School, 1969.

SQUARES INSCRIBED IN RIGHT TRIANGLES

GRADE LEVEL/ COURSE: Geometry

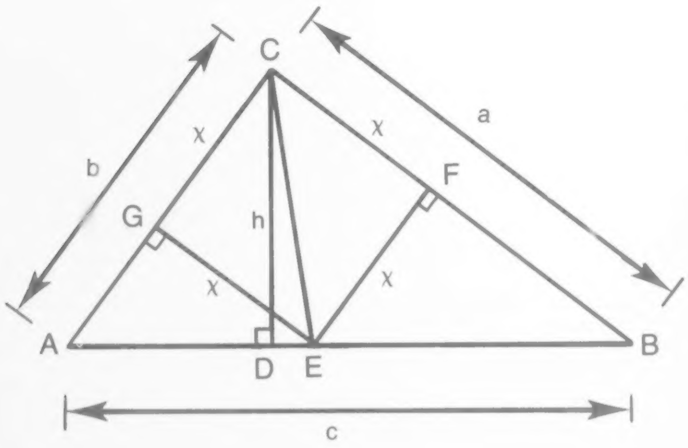
INTRODUCTION/ OBJECTIVE: The students will translate a word problem involving the inscribed square of a right triangle into a geometric diagram. The students will identify similar triangles and use the ratios associated with these similar triangles.

PROCEDURE: The teacher will present the problems to the whole class and ask them to generate the diagrams and name similar triangles either individually or in small groups. Solutions should be discussed in class.

ACTIVITY 1: **Step 1.** Give the students the following problem to consider:

Given a right triangle with hypotenuse c and altitude to the hypotenuse h, express in terms of c and h the length of the side of an inscribed square which lies on the legs of the triangle.

Step 2. Each student should draw a right triangle ABC with C at the right angle. Draw $\overline{CE}$ bisecting $\angle C$. From E on $\overline{AB}$, draw $\overline{EF} \perp \overline{CB}$ and $\overline{EG} \perp \overline{AC}$. Label $\overline{AC}$ as b, $\overline{BC}$ as a, and $\overline{AB}$ as c. Draw $\overline{CD} \perp \overline{AB}$ and label $\overline{CD}$ as h. Let x represent the length CF. Teachers should check each student's drawing and labeling for accuracy before proceeding to Step 3.



Step 3. Ask the students to identify similar triangles. ($\triangle AGE$ and $\triangle ACB$ are similar triangles.) From the study of similar triangles,

$$\frac{GE}{CB} = \frac{AG}{AC}$$

Step 4. Point out from the diagram that $GE = x$, $CB = a$, $AG = b - x$ and $AC = b$. Ask the students to use these equivalencies; to rewrite the given ratio

$$\frac{GE}{CB} = \frac{AG}{AC}$$

(Solution: $\frac{x}{a} = \frac{b - x}{b}$)

Step 5. Have the students solve this proportion for x.

$$\frac{x}{a} = \frac{b - x}{b}$$

$$bx = a(b - x)$$
$$bx = ab - ax$$
$$bx + ax = ab$$
$$(a + b)\,x = ab$$

$$\boxed{x = \frac{ab}{a + b}}$$

Step 6. Point out that they have now solved for x, the length of a side of the inscribed square. They now need to express x in terms of c and h, instead of a and b. Ask students to give the formula for the area of the triangle: ($A = \frac{1}{2}bh$ where b is the base and h is the height.) The area of $\triangle ABC = \frac{1}{2}ab$ or
$\triangle ABC = \frac{1}{2}ch$

Have the student point out why both of these statements are true by using the diagram. Since both are true, point out that $\frac{1}{2}ab = \frac{1}{2}ch$; therefore, twice the area of $\triangle ABC = ab$ which equals ch.

Step 7. Beginning with $a + b = \sqrt{(a + b)^2}$ and substitutions from above, show that $a + b = \sqrt{c^2 + 2ch}$ as follows:

$$a + b = \sqrt{(a + b)^2}$$
$$a + b = \sqrt{(a^2 + 2ab + b^2}\text{ but } ab = ch$$
$$\text{so, } a + b = \sqrt{a^2 + 2ch + b^2}\text{ but } a^2 + b^2 = c^2 \text{ (Pythagorean Theorem)}$$
$$\text{so, } a + b = \sqrt{c^2 + 2ch}$$

Step 8. Since $x = \frac{ab}{a + b}$ (Step 5) and

$ab = ch$ (Step 6) and

$a + b = \sqrt{c^2 + 2ch}$ (Step 7)

then, $x = \frac{ch}{\sqrt{c^2 + 2ch}}$

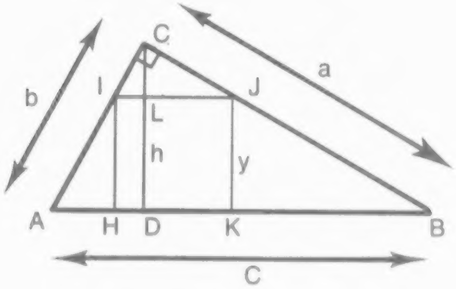
ACTIVITY 2:

Step 1. Give the students the following problem to consider:

Given the same information as in Activity 1, express in terms of c and h the length of a side of the inscribed square resting on the hypotenuse.

Step 2. Instruct the students to draw a diagram of the given problem labeling the $\triangle ABC$. Tell them to let a, b, c, and y represent the lengths of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ and the side of the inscribed square HIJK, respectively. Let CD be the altitude to $\overline{AB}$ and represent its distance by h.

Step 3. Check each students diagram by the following diagram as an example of a correctly labeled diagram.



Step 4. Ask the students to name some similar triangles in the diagram. These should include

$$\triangle ICJ \sim \triangle ACB$$
$$\triangle ICL \sim \triangle ACD$$

These similarities prove $\frac{IJ}{AB} = \frac{CL}{CD}$

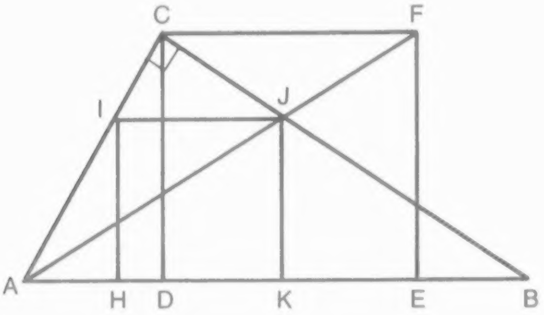
Since CD is represented by h, CL may be represented by h − y, and the following substitutions may be made

$$\frac{y}{c} = \frac{h - y}{h}$$

Have the students solve this proportion for y, as follows:

$$\frac{y}{c} = \frac{h - y}{h}$$
$$yh = c(h - y)$$
$$yh = ch - cy$$
$$yh + cy = ch$$
$$y(h + y) = ch$$
$$\boxed{y = \frac{ch}{h + c}}$$

EXTENDED LEARNING: A second method may be used to solve the problem in Activity 2. The initial diagram would be the same as in Step 3 of Activity 2. Then a second square must be constructed with CD as a side. AF must be drawn intersecting CB in J.



From the diagram, the following triangles are similar △AIH ~ △ACD, △AIJ ~ △ACF and △ICJ ~ △ACB: therefore the following ratios are equal

$$\frac{HI}{CD} = \frac{AI}{AC} = \frac{IJ}{CF} \text{ and } CD = CF$$

It follows that HI = IJ and HIJK is a square.

Considering the fact that △ICJ ~ △ACB and △AIJ ~ △ACF it can be shown that $\frac{1}{y} = \frac{1}{h} = \frac{1}{c}$ if $h = CD$, $y = HI$ and $c = AB$. To reach this conclusion we will let $CI = p$ and $IA = q$. Because of the similar triangle

$$\frac{q}{y} = \frac{p+q}{h} \text{ and } \frac{p}{y} = \frac{p+q}{c}$$

$$\frac{q}{y} + \frac{p}{y} = \frac{p+q}{h} + \frac{p+q}{c}$$

$$\frac{p+q}{y} = \frac{p+q}{h} + \frac{p+q}{c}$$

Multiplying by $\frac{1}{p+q}$, $\frac{1}{y} = \frac{1}{h} + \frac{1}{c}$

If we solve for y then $y = \frac{ch}{c+h}$

This shows that the side of the inscribed square is ½ the geometric mean between the hypotenuse and the altitude to the hypotenuse.

RESOURCE: Unpublished paper by Evans Anuforoh.

SQUARING A NUMBER ENDING IN FIVE

GRADE LEVEL/ COURSE: 4–6

INTRODUCTION/ OBJECTIVE: Students will be able to square any number ending in 5, without the use of standard multiplication.

PROCEDURE: Prior to this lesson, students should understand the concept of squaring a number.

ACTIVITY:

Step 1. Have the students square each of these by conventional means:

$75^2 \qquad 115^2$

Step 2. After the students have completed the above computations, they will probably welcome an easier solution to these problems.

To square any number ending in 5, multiply the numbers to the left of the 5 by the next higher integer and then attach 25. For example, 75^2:

$$\begin{array}[b]{r}75\\ \times\ 75\\ \hline 5625\end{array}\qquad \begin{array}[b]{r}8\\ \times\ 7\\ \hline 56\end{array}\ \text{ and }\qquad \begin{array}[b]{r}5\\ \times\ 5\\ \hline 25\end{array} \rightarrow 5625$$

The number to the left of 5 is 7. Multiply 7 by the next higher integer (7 + 1 = 8). This gives us 56 to which we attach a 25. Thus 75^2 is 5625.

Step 3. Now, illustrate another example. Find the square of 115.

$$\begin{array}[b]{r}115\\ \times\ 115\\ \hline 13225\end{array}\qquad \begin{array}[b]{r}12\\ \times\ 11\\ \hline 132\end{array}\ \text{ and }\qquad \begin{array}[b]{r}5\\ \times\ 5\\ \hline 25\end{array} \rightarrow 13225$$

The number to the left of the 5 is 11. Multiply 11 by the next higher integer (11 + 1 = 12). This gives us 132. We attach a 25. Thus 115^2 is 13225.

Step 4. Have the students square 35 using the short cut method.

$$\begin{array}[b]{r}35\\ \times\ 35\\ \hline 1225\end{array}\qquad \begin{array}[b]{r}4\\ \times\ 3\\ \hline 12\end{array}\ \text{ and }\qquad \begin{array}[b]{r}5\\ \times\ 5\\ \hline 25\end{array} \rightarrow 1225$$

Step 5. Have the students square these numbers using the short cut method. Then have them check their answers using the conventional method.

$65^2 \qquad 995^2$

EXTENDED LEARNING: Have the students develop an algebraic solution for the short cut method.

REFERENCE: Laderman, Jack. *Short Cuts in Arithmetic*. New York: Jayel Publishing Company, 1936.

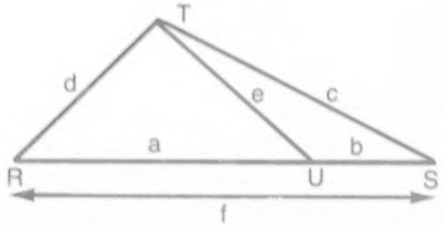
STEWART'S THEOREM

GRADE LEVEL/ COURSE: Geometry. The student should know the properties of medians and altitudes of triangles, be able to use the Pythagorean Formula, and simplify radical expressions.

INTRODUCTION/ OBJECTIVE: The student will find the length of a line segment joining the vertex of a triangle and a point on the opposite side.

HISTORY: A line from a vertex of a triangle to the opposite side is called a cevian. The altitude and median of a triangle are special kinds of cevians. "Cevian" is named after Giovanni Ceva. Matthew Stewart (1717–1785) was a Scottish mathematician who developed a formula for determining the length of a cevian of a triangle. His formula states that:

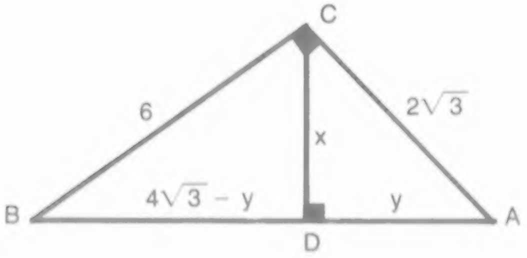
$$d^2b + c^2a = f(e^2 + ab)$$



PROCEDURE:

1. Define cevian.
2. Sketch several triangles and label the altitude, median, angle bisector, and a general cevian in each.
3. Ask how one can determine the measure of the cevian. (Possible answers might be Pythagorean Theorem, trigonometric formulas, etc.). Explain the history and use of Stewart's Theorem.
4. The students will be expected to find the length of a cevian, whether it is an altitude, median, or just a general cevian.
5. Have the students draw and label a triangle with a cevian.
6. Ask the students to write an equation using Stewart's Theorem relating it to their diagram.
7. Have the students put numbers on all line segments on the diagram except the cevian (check to be sure all are valid triangles, (i.e., the sum of the lengths of any two sides must exceed the length of the third side.)
8. Using Stewart's Theorem, find the length of the cevian.
9. For additional emphasis, have the students break into groups of three and work more examples.

ACTIVITY 1: Use first the Pythagorean Theorem and then Stewart's Theorem to solve for a cevian, in the following right triangle.



1) Pythagorean Theorem Method

a) $AD^2 + CD^2 = AC^2$ can be written $x^2 + y^2 = (2\sqrt{3})^2$ using the variables indicated.

b) $CD^2 + DB^2 = BC^2$ can be written $x^2 + (4\sqrt{3} - y)^2 = 6^2$ using the variables indicated.

Now subtract equation (b) from equation (a) to get an equation in one variable.

$$\begin{array}{r} x^2 + y^2 = 12 \\ -(x^2 + (4\sqrt{3} - y)^2 = 36) \\ \hline y^2 + (4\sqrt{3} - y)^2 = 12 - 36 \\ y^2 - 48 + 8y\sqrt{3} - y^2 = -24 \end{array}$$

$$y = {}^{24}/8\sqrt{3} = {}^{3}/\sqrt{3} = \frac{3}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{3\sqrt{3}}{\sqrt{9}} = \sqrt{3}$$

$$\text{thus } y = \sqrt{3}$$

Now substituting in equation (a) and solving for x we have

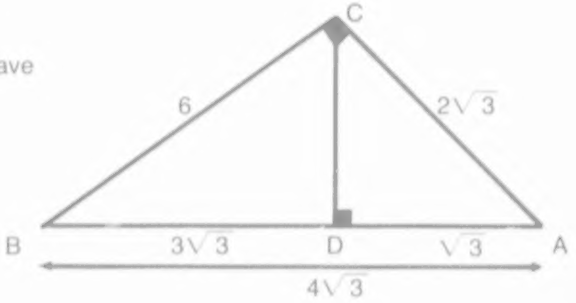
$$x^2 + (\sqrt{3})^2 = (2\sqrt{3})^2$$

$$x^2 = 9$$

$$x = 3$$

$$\therefore CD = 3$$

The length of the cevian is 3.



2) Stewart's Theorem Method

$$BC^2 \cdot DA + AC^2 \cdot BD = AB\,(CD^2 + BD \cdot AD)$$

$$6^2 \cdot \sqrt{3} + (2\sqrt{3})^2 \cdot 3\sqrt{3} = 4\sqrt{3}(X^2 + 3\sqrt{3} \cdot \sqrt{3})$$

$$36\sqrt{3} + 36\sqrt{3} = 4\sqrt{3}(X^2 + 9)$$

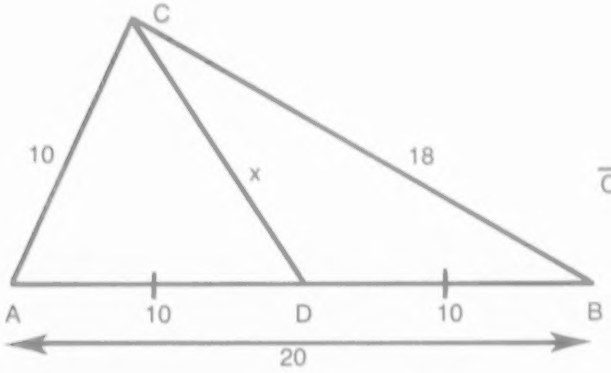
$$X^2 + 9 = \frac{72\sqrt{3}}{4\sqrt{3}} = 18$$

$$X^2 = 9$$

$$X = 3$$

$$\therefore CD = 3$$

ACTIVITY 2: Use Stewart's Theorem to find the length of a median of a triangle. The lengths of the sides were chosen arbitrarily. (Note: 10, 18, 20 meet the test for triangularity, 10 + 18 > 20).



$\overline{CD}$ is a median

$$AC^2 \cdot DB + BC^2 \cdot AD = AB\,(CD^2 + AD \cdot DB)$$

$$10^2 \cdot 10 + 18^2 \cdot 10 = 20(x^2 + 10 \cdot 10)$$

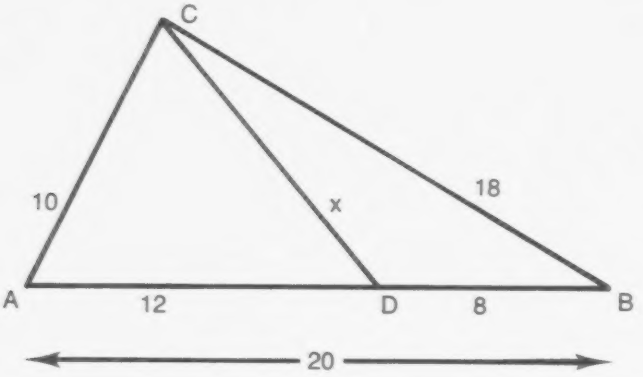
$$\frac{1000 + 3240}{20} = x^2 + 100$$

$$x^2 = 112$$

$$x = 4\sqrt{7}$$

$$\therefore CD = 4\sqrt{7}$$

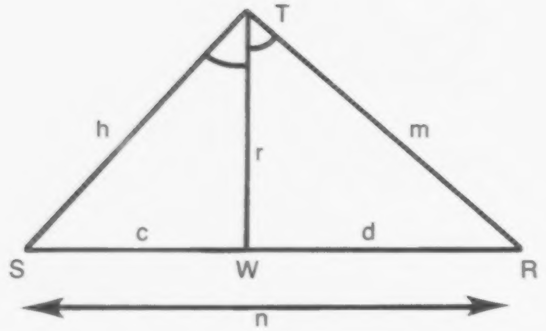
ACTIVITY 3: Use Stewart's Theorem to find the length of any cevian:



$\overline{CD}$ is a Cevian

$$\begin{aligned}
AC^2 \cdot DB + BC^2 \cdot AD &= AB\,(CD^2 + AD \cdot DB) \\
10^2 \cdot 8 + 18^2 \cdot 12 &= 20(x^2 + 12 \cdot 8) \\
\frac{800 + 3888}{20} &= x^2 + 96 \\
x^2 + 96 &= 234.4 \\
x^2 &= 138.4 \\
x &\approx 11.76
\end{aligned}$$

EXTENDED LEARNING: 1.



In $\triangle RST$, $\overline{TW}$ is the angle bisector of $\angle RTS$. Show that Stewart's Theorem can be written as $r^2 = hm - cd$ when r is the length of the angle bisector shown in the figure. Then find r when $h = 9$, $m = 15$, $c = 6$, and $d = 10$.

2. In an isosceles triangle with two sides of measure 17, a line measuring 16 is drawn from the vertex to the base. If one segment of the base, as cut by this line, exceeds the other by 8, find the measure of the two segments.

REFERENCES:

Posamentier, Alfred S. and Sheridan, Gordon. *Math Motivator: Investigations in Geometry, Algebra, Pre-Algebra*. Menlo Park, CA: Addison-Wesley, 1982, 1983.

SUM OF THE ANGLE MEASURES OF POLYGONS

GRADE LEVEL/ COURSE: 4–6

MATERIALS: Several sheets of paper, pencil, scissors, straight edge, transparent tape.

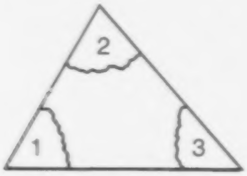
INTRODUCTION/ OBJECTIVE: The student will express the sum of the angle measures of a polygon as a number of straight angles.

PROCEDURE: This activity may be done as a whole class activity where the students work in pairs. The teacher will need to give careful directions.

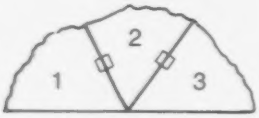
ACTIVITY:

Step 1. Have students use a straight edge to draw any kind of triangle. Students may need to be encouraged to draw straight line segments.

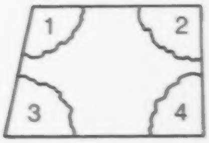
Cut out the triangle.

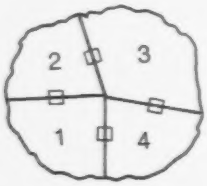


Step 2. Have the students tear off each angle of the triangle. See diagram. Students should notice from this that the sum of the angles of a triangle will be equivalent to a straight angle.

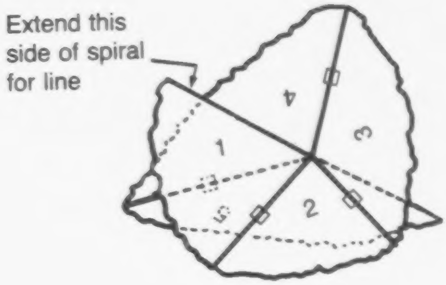


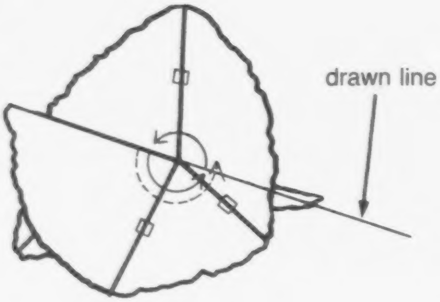
Step 3. Repeat the above activity using an *irregular* quadrilateral. See diagram below. Students will see that the sum of the angles of a quadrilateral are equivalent to two straight angles (360°).





Step 4. Repeat the procedure using an irregular pentagon. When the angles are taped together a sort of spiral arrangement (over-lapping) results. Begin with the lash edge of the spiral and draw a straight line. Mark the 180° angle between the spiral edge and the line. Mark the next 180° angle. Now lift the lash part of the spiral and mark the last angle which should be 180°. Students will see from this that the sum of the interior angles of the pentagon is equivalent to three straight angles (3 × 180 = 540).





EXTENDED LEARNING: Continue this activity for polygons of six, seven, eight, and more sides. Stress to your students the importance of accurate construction to avoid errors.

SUM OF N ODD NUMBERS

GRADE LEVEL/ COURSE: 7–8

INTRODUCTION/ OBJECTIVE: The student will write a simple formula for the pattern observed when finding the sum of n odd numbers.

PROCEDURE: This activity may be used when students are working with formulas or when working with exponents. This activity should encourage students to reach conclusions by analyzing number patterns.

ACTIVITY 1: Have students use the formula $\frac{n(n + 1)}{2}$ for finding the sum of the first n whole numbers.

ACTIVITY 2: Have students experiment with sums of the first n odd number as follows:

Number of odd numbers		sum
1	1	or 1
2	1 + 3	or 4
3	1 + 3 + 5	or 9
4	1 + 3 + 5 + 7	or 16
5	1 + 3 + 5 + 7 + 9	or 25
•	•	•
•	•	•
•	•	•

ACTIVITY 3: Have students look for a pattern in the sums. They should conclude that the sums of the odd numbers are the squares of the numbers in the first column.

ACTIVITY 4: Have students express this relationship as a formula. ($n \times n$ or, n^2)

EXTENDED LEARNING: Students may experiment to see if a formula for the first n *even* numbers can be written. The formula is $n(n - 1)$. Remember that the first even whole number must be zero.

SYMMETRY OPERATIONS

GRADE LEVEL/ COURSE: 6–8

MATERIALS: Mirror, cut-out of the letters A, E, & N, and cut-out of an equilateral triangle.

INTRODUCTION/ OBJECTIVE: This activity will develop the idea of symmetry operations. Students will be able to define reflections and rotations, identify these operations for letters of the alphabet and transfer the concept to geometric figures.

PROCEDURE: This activity can be done with the entire class or with a small group. Finding the symmetry of alphabet letters can be an outside assignment.

ACTIVITY: Begin by asking students what they see when they look in a mirror. They will say, of course, that they see themselves. Hold up a mirror and place the letter A in front of the mirror. Students will notice that the reflection of the letter A in the mirror is the letter A (fig. 1)

Repeat this experiement with the letter E. Students will quickly notice that the reflection of E looks like this Ǝ (fig. 2). Repeat with the letter N. Again, N is reflected as its reverse (fig. 3).

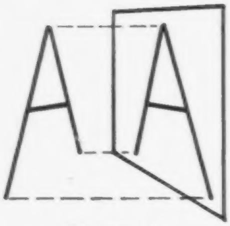


Figure 1

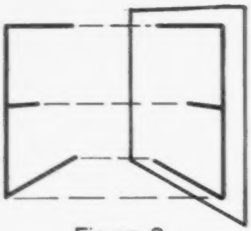


Figure 2

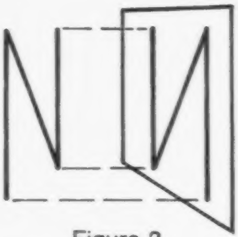
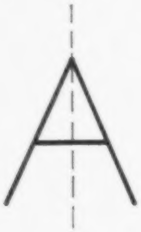


Figure 3

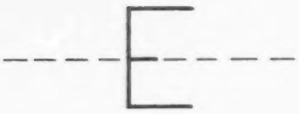
Stand in the middle of the class and hold up the A. The A will look like an A to all students, those before and behind you. The E and the N will not pass this test, though. The students behind you will say the letters are reversed, or backwards.

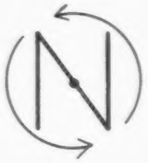
State that the reason A passes this test is because it has what is called vertical symmetry. Demonstrate on the blackboard with the following diagram:



Place the cutout A upon the diagram and flip it about the dotted vertical line. Each flip results in the letter A. Flipping the E and the N in this way gives reversed letters.

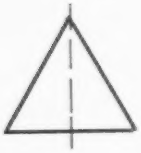
Elicit the solution for the letter E (i.e., that E must be flipped about a horizontal axis). The letter E has horizontal symmetry. It can be easily be shown that the letter N has neither vertical nor horizontal symmetry. Students may suggest two solutions. One, that N be flipped *both* vertically and horizontally, in either order, and two, that N be rotated 180°, i.e., not lifted off its plane, but pivoted (reflected in a point) on a central point by a half-turn.

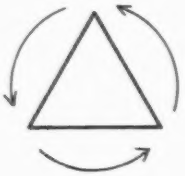




We have so far demonstrated that certain objects (letters) have symmetry. Further, we have demonstrated symmetry by using flips and rotations. We call these flips, rotations, and combinations of them, symmetry operations. They can be applied to the following game.

Begin by showing that a cut-out equilateral triangle may be flipped about any of its angle bisectors (choose the vertical). Also, the triangle may be rotated, i.e., turned counter-clockwise 120° (why 120°?)





Ask for teams of volunteers consisting of three students each, to come before the class to demonstrate various moves called out by the rest of the class. For example, with students Anne, Bill, and Cathy positioned (fig. 4),

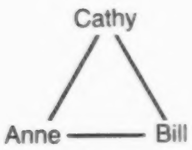


Fig. 4

the class calling for a flip requires a new position. (fig. 5)

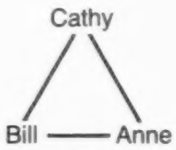
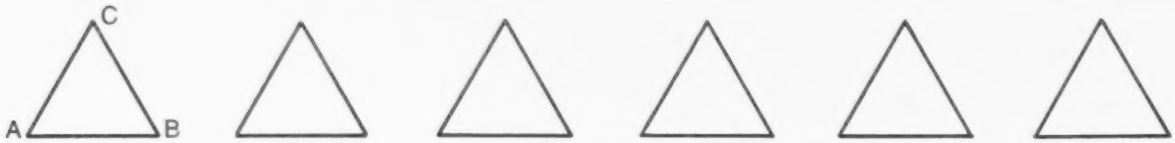


Fig. 5

As students become more proficient, two moves may be called at once.

Check the symmetry operations for the other 23 letters of the alphabet. Are there some letters without symmetry? Are there letters with more than one symmetry operation?

EXTENDED LEARNING: Given triangle ABC below, re-arrange the letters A, B, C, at different vertices to get five more (total of six) *different* arrangements (permutations).



Make a list of the symmetry operations that will produce each new arrangement in the above. (Students familiar with the dihedral group may find that this last exercise gives the six elements of D_3.)

REFERENCES:

Coxeter, H. M. S. *Introduction to Geometry*. New York: John Weley and Sons, Inc., 1961.

Dienes, Z. P. and Golding, E. W. *Geometry Through Transformation*. New York: Herder and Herder, 1967.

TAXI-CAB DISTANCES

GRADE LEVEL/ COURSE: Geometry

MATERIALS: Paper, pencils, graph paper, copies of Figure 1.

INTRODUCTION/ OBJECTIVE: Given a non-traditional definition of 'distance', students will analyze distance problems and compare the results with traditional measures of distance.

PROCEDURE: Students should be familiar with applications of distance in coordinate geometry. In addition, they should be familiar with various kinds of locus problems. Work with the entire class in small groups on this activity.

Have the students state the distance formula: $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ = distance between x and y.

Introduce the concept of taxi-cab distance as the path a cab can travel following horizontal or vertical lines only. (the roads are designated by the lines on the grid — see Figure 1).

For example, looking at the grid, we are given two points A (x_1, y_1) and B (x_2, y_2). The different paths indicate that we can travel from point A to point B in a variety of ways. The concept of the 'shortest' distance between 2 points being the straight line connecting A and B doesn't apply here. Taxi-cabs can't drive through buildings. Regardless of the path here, our taxi-cab distance is 9.

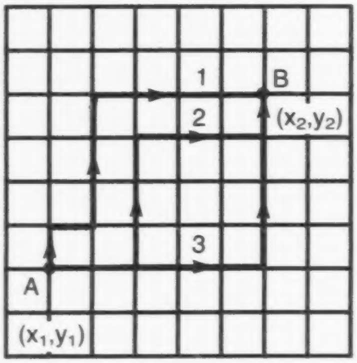


Figure 1

Now ask students to develop a formula for this taxi-cab distance. $TD = x_2-x_1 + y_2-y_1$. The meaning of this formula basically indicates that taxi-cab distance is the sum of all vertical and horizontal movement between any two given points. (Attempt to have the class derive the formula and explanation.)

Study variations between 'standard distance' and taxi-cab distance. Use the following chart as a guide.

	P_1	P_2	Standard Distance	Taxi-cab Distance
	(4, 7)	(5, −1)	65	9
N_1	(0, 2)	(−7, 2)	7	7
	(−1, −6)	(1, −4)	8	4
N_2	(8, −5)	(8, 0)	5	5
	(3.4, 1.7)	(2.5, 2.9)	1.5	2.1
	(0, 0)	(a, b)	$\sqrt{a^2 + b^2}$	$\|a\| + \|b\|$
N_3	(a, b)	(c, d)	$\sqrt{(a-c)^2 + (b-d)^2}$	$\|a-c\| + \|b-d\|$

When will the standard distance equal the taxi-cab distance? N_1 and N_2 help us see that when the 2 points can be connected by a vertical or horizontal line, standard distance equals taxi-cab distance.

Show this using the general case N_3.

Note: $\sqrt{x^2} = |x|$ and $x^2 = |x|^2$

$$\sqrt{(a-c)^2 + (b-d)^2} = |a-c| + |b-d|$$

$$(a-c)^2 + (b-d)^2 = |a-c|^2 + |b-d|^2 + 2\,|a-c|\,|b-d|$$

$$0 = 2\,|a-c|\,|b-d|$$

Therefore, the standard distance will be equal to the taxi-cab distance only if $a-c = 0$ or $b-d = 0$.

This confirms that the two points can be connected by a vertical or horizontal line on the grid.

ACTIVITY 2: Find the locus of points 2 units away from a given point 'P'. What would normally be a circle of radius 2 becomes a collection of points indicating the shape of a square. If we regard this shape as a square for a moment, one can observe that using taxi-cab distance, the distance AB is equivalent to the distance BC.

Extending this further we can see that the theorem in Euclidean Geometry stating that 'the sum of any 2 sides of a given triangle must exceed the third' does not hold true in taxi-cab geometry.

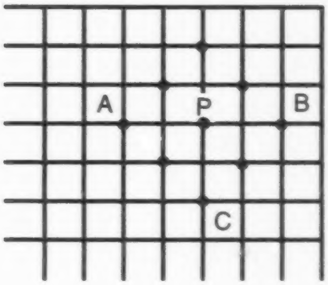
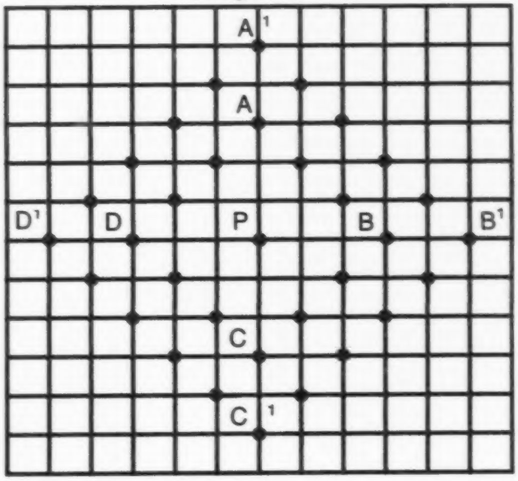


Figure 2

ACTIVITY 3: Find all points whose taxi-cab distance from point 'P' (located on the intersection of two streets) is 3, 5, and a (where $a \in N$)

Figure 3



Since ABCD and $A^1B^1C^1D^1$ are square-like shapes, let's call DB and D^1 B^1 diagonals of their respective square-like-shapes (S.L.S.) and AD and A^1 D^1 the sides. With this understanding, complete the following chart using taxi-cab distance.

Distance from Point P	TD of Diagonals	TD of Sides
(fig. 2) 2	4	4
(fig. 3) 3	6	6
(fig. 3) 5	10	10
(fig. 3) a	2a	2a

(chart to be left blank for students.)

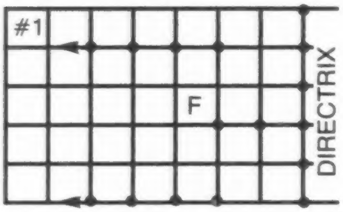
By following the pattern shown, we can see that the locus of points 'a' distance from point P is another S.L.S. The diagonals and sides (as we redefined them) will equal '2a' and the diagonals will intersect at point P.

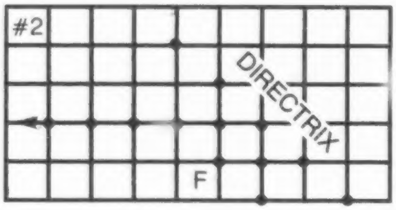
ACTIVITY 4: To adapt parabolas to taxi-cab geometry, we need to redefine directrix.

Directrix: 'a set of points' from the Euclidean line (in Euclidean geometry the directrix is a straight line).

For each of the following, given a directrix and a focus 'F', graph the locus of points whose distance to the focus is equal to the shortest distance to the directrix.

Shortest distance to the directrix.





EXTENDED LEARNING:

1. The square-like shape in taxi-cab geometry replaces the circle from Euclidean geometry. Have students show that the equivalent of π in taxi-cab geometry is 4.
2. Find the locus of points equidistant from two points that are:
   (a) on the same street
   (b) on different streets
3. Do taxi-cab geometry in three dimensions.

REFERENCES:

Franco, Anna. "Taxi-cab Geometry". *Math Survey.* New York: Stuyvesant High School, 1982.

Gardner, Martin. "Mathematical Games". *Scientific American* November, 1980.

TOPOLOGY (EQUIVALENCE)

GRADE LEVEL/ COURSE: 6–8, Geometry

MATERIALS: Modeling clay, balloons, rubber band, graph paper.

INTRODUCTION/ OBJECTIVE: The students will classify figures as topographically equivalent and state the property for each classification. Topology is concerned with objects in space: their points, lines, shapes, changes that are continuous, and properties that are affected by changes in shape or size. This field of mathematics was developed in the late 19th century. Topologically equivalent figures are grouped according to their properties.

a. The figure remains a simple closed figure.

b. The figure has one hole (genus 1), two holes (genus 2), three holes (genus 3).

c. The figure is a line.

d. The figure is divided into an equal number of regions with one point of intersection.

e. Points on the figure remain in order.

f. On solid figures every closed curve can be contracted to a point without passing out of the surface.

If a figure can be bent, stretched or molded without being cut or punctured, into another figure, the two are topographically equivalent.

PROCEDURE: The teacher may use the following activities with large groups, small groups, or individuals to allow the students the opportunity to discover the properties of equivalence listed in the introduction. These activities can be completed in 2–3 class periods.

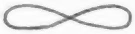
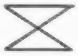
ACTIVITY 1: Make the statement which follows to the students. Topologists believe that:

a. a square = triangle = a circle

b. "B" = eyeglasses

c. doughnut = coffee cup

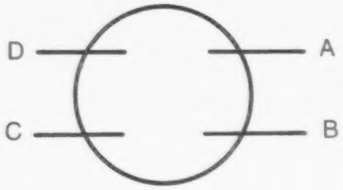
d. solid ball = block = potato

e.  = 

Do you agree with the topologists? Explain.

ACTIVITY 2: Give the students a balloon. Have them draw a circle around the center of the balloon. Blow the balloon up. What changed? (line stretched and became distorted, balloon stretched) What remained the same? (balloon still a closed curve, line remained in the same place)

ACTIVITY 3: Have students take a rubber band, four small pieces of masking tape and write letters A, B, C, D, on the tape. Attach the tape to the rubber band in this order:

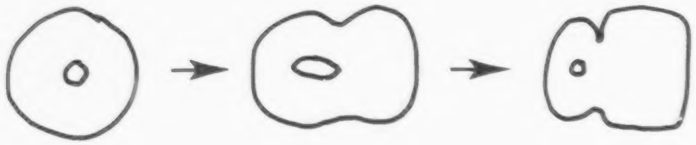


Students stretch the rubber band to change from a circle to a square to a triangle. What properties changed? (shape) What properties remained the same? (remained a closed curve, points remained in order.)

ACTIVITY 4: On a sheet of paper have the students draw a number line from 1 to 5, then crumble the paper and open but do not smooth. What changed? What remained the same?

ACTIVITY 5: Have the students draw a line on a paper. Now see how many letters of the alphabet (upper case) that this line could be bent or twisted to make. Name them.

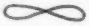
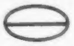
ACTIVITY 6: Have the students take modeling clay and form into a doughnut. Now stretch to form a coffee cup.



What changed? (shape) Remained the same? (one hole)

ACTIVITY 7: Have the students name other objects that would be equivalent to a doughnut (genus 1). (ring, triangle, funnel)

ACTIVITY 8: Do you agree now that eyeglasses and "B" are equivalent? Why? (2 holes, genus 2)

ACTIVITY 9: How is  like  ? (two regions) Different? (two points of intersection) Take a rubber band and see if it can be bent or stretched to make the both shapes. (No) What must the property be? (have same number of regions and *one* point of intersection)

ACTIVITY 10: Have the students take a piece of clay and form a solid ball. Change from a solid ball to a solid block. What changed? (size, shape) What remained the same? (every closed curve drawn on it can be contracted to a point without passing out of the surface)

ACTIVITY 11: Have students go back to Activity 1. and answer the question again. Agree or disagree? Why?

ACTIVITY 12: Have students group these block letters into topographically equivalent groups and state the property of each: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. (straight line — 1, 2, 3, 7; genus 1 [one hole] — 0, 4, 6, 9; genus 2 [two holes] — 8)

ACTIVITY 13: Now take the upper case letters of the alphabet and group into topographically equivalent groups and state their properties. (straight line — C, I, L, M, N, V, W, S: genus 1 — D, O: genus 2 — B)

EXTENDED LEARNING:

Graph a figure with the following points:

x	y
1	5
2	4
9	11
9	13
10	12
10	10
12	10
13	9
11	9
4	2
5	1
1	1
1	5

Change the y values to 2y. What changed? Remained the same?

REFERENCES:

Posamentier, Alfred S. and Sheridan, Gordon. *Math Motivators! Investigations in Pre-Algebra.* Reading, Mass.: Addison Wesley Publishing Company, 1982.

Burns, Marilyn. *The I Hate Mathematics! Book.* Boston: Little, Brown and Company, 1975.

Jacobs, Harold R. *Mathematics: A Human Endeavor.* New York: W. H. Freeman and Company, 1982.

TRANSFORMATIONS WITH ORDERED PAIRS

GRADE LEVEL/ COURSE: 5–6

MATERIALS: Graph paper with preferably 1 cm squares, one paper strip.

INTRODUCTION/ OBJECTIVE: This activity offers an opportunity for students who have mastered rectangular coordinates to apply the concept.

PROCEDURE: This is an activity for all students to start in class and finish for homework. An example should be demonstrated by the teacher.

Students will draw a figure on graph paper in the first quadrant. They will list the ordered pairs which define the figure on a paper strip. The lists will be exchanged and for homework students will redraw the picture in a new position. (See example). The position change will be restricted by the original drawing and the size of the paper. The next day students will compare their drawings to the original.

ACTIVITY:

Step 1. Give the students the following set of materials. The sets should be numbered individually for each student. (e.g., All of John's papers will be numbered 4, all of Susan's papers will be numbered 7, etc.)

one section grid paper
one section grid paper with a paper strip attached

Step 2. Direct the students to draw and label a pair of axes on a single section of grid paper. Then draw a simple figure or picture of straight lines, labeling the ordered pairs which define the figure in order.

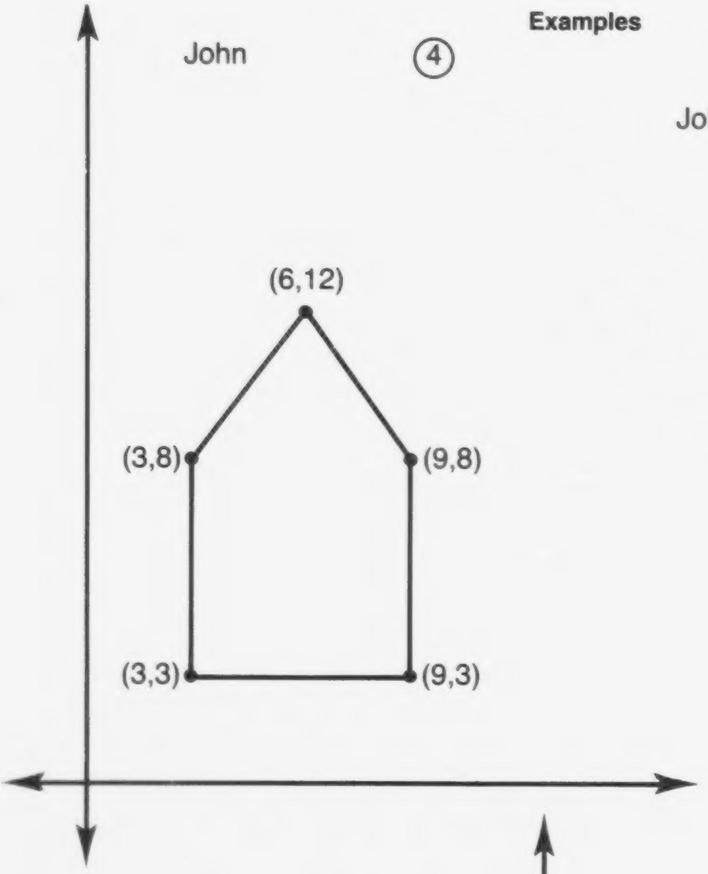
Step 3. Each student then lists the ordered pairs for his picture on the paper strip.

Step 4. Take up only the pictures. Then have pairs of students exchange the strips attached to the blank grid.

Step 5. For homework, students will examine the ordered pairs and determine how they can draw the picture in a new position. They are to draw and label the axes, reconstruct the picture in its new position, label the new set of ordered pairs, and describe the transformation.

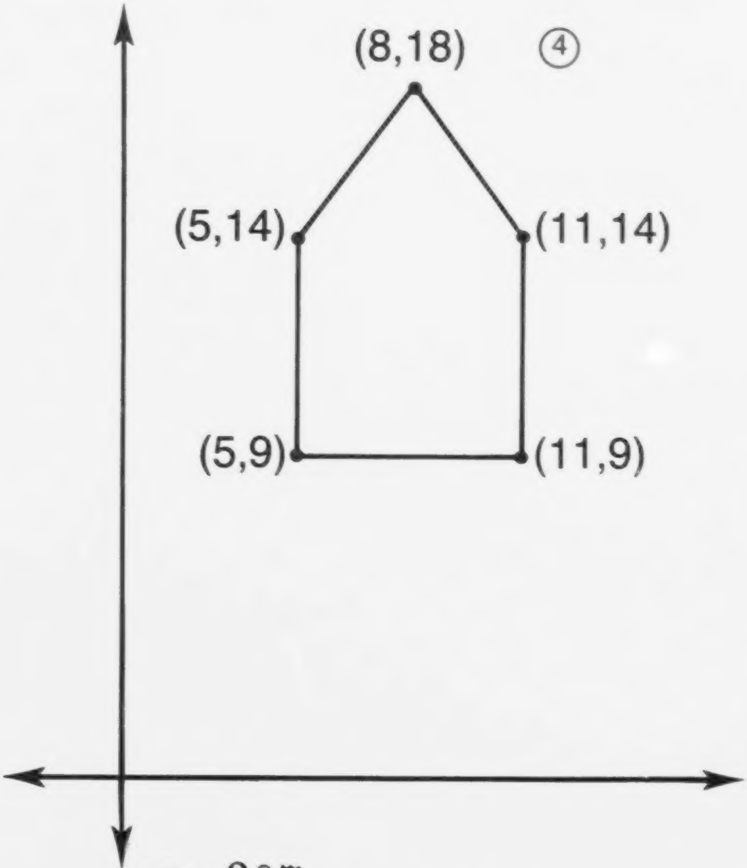
Step 6. The next day redistribute the original pictures and have the students compare the homework drawings to the originals. They should discuss any difficulties they had.

Examples



John ④

(3,3)
(9,3)
(9,8)
(6,12)
(3,8)



over 2, up 6

TRIANGULAR NUMBERS

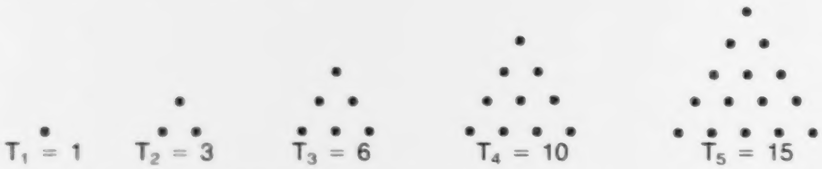
GRADE LEVEL/ COURSE: 7–8

INTRODUCTION/ OBJECTIVE: The student will represent triangular numbers with dots and find a formula for triangular numbers. The student will find a pattern when adding consecutive natural numbers and consecutive triangular numbers.

HISTORY: Figurate or polygonal numbers originated about 500 BC with the earliest members of the Pythagorean brotherhood. Figurate numbers, represented by dots in certain geometrical configurations, are a link between geometry and arithmetic. Triangular numbers can be represented by an array of objects in the shape of an equilateral triangle.

PROCEDURE: The following are teacher directed activities which may be used with large or small groups. The extended activities may be used as homework.

ACTIVITY: **Step 1.** Show the first 5 triangular numbers represented as dots.



Step 2. Have students represent the next 5 triangular numbers as dots.

Step 3. Have students find the pattern of triangular numbers

$0_{+1}\quad 1_{+2}\quad 3_{+3}\quad 6_{+4}\quad 10_{+5}\quad 15_{+6}\quad \text{—}_{+7}$

Step 4. Help students find a formula or rule for T_n in terms of n.

$$T_n = \frac{n(n+1)}{2}$$

Step 5. Using the formula, list the first 15 triangular numbers.

Step 6. Have students find the sums of the consecutive natural numbers.

$$\begin{aligned}
&1 &&= 1\\
&1 + 2 &&= 3\\
&1 + 2 + 3 &&= 6\\
&1 + 2 + 3 + 4 &&= 10\\
&1 + 2 + 3 + 4 + 5 &&= 15\\
&1 + 2 + 3 + 4 + 5 + 6 &&= 21\\
&1 + 2 + 3 + 4 + 5 + 6 + 7 &&= 28
\end{aligned}$$

Note the pattern.
(Each sum of consecutive natural numbers is a triangular number.)

Step 7. Have students find the sum of 2 consecutive triangular numbers.

$$1 + 3 = 4 = 2^2$$

$$3 + 6 = 9 = 3^2$$

$$6 + 10 = 16 = 4^2$$

$$10 + 15 = 25 = 5^2$$

$$15 + 21 = 36 = 6^2$$

Note the pattern.
(The sum of 2 consecutive numbers is a perfect square.)

EXTENDED LEARNING:

1. Draw some dot arrays for square, pentagonal, hexagonal and other polygonal numbers.
2. Fill in a chart and study the patterns horizontally and vertically.

Number	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Triangular	1	3	6	10	15	21	28	36	45	55
Square	1	4	9	16	25	36	49	64	81	100
Pentagonal	1	5	12	22	35	51	70	92	117	145
Hexagonal	1	6	15	28	45	66	91	120	153	190
Heptagonal	1	7	18	34	55	81	112	148	189	235
Octagonal	1	8	21	40	65	96	133	176	225	280
Nonagonal	1	9	24	46	75	111	154	204	261	325
Decagonal	1	10	27	52	85	126	175	232	297	370
Vertical Pattern		+1	+3	+6	+10	+15	+21	+28	+36	+45

THE UNFINISHED BUSINESS OF MATHEMATICS

GRADE LEVEL/ COURSE: Algebra II

INTRODUCTION/ OBJECTIVE: This activity will demonstrate that our system of mathematics is not complete, but needs further study.

Students sometimes get the impression that there is no more "new ground" in mathematics; that everything is already known and has been set down — the student's job being only to learn that given body of knowledge the same as all the generations that have preceded them. It may, therefore, be surprising to students to learn that there are fairly straightforward and easily understandable problems in mathematics that have nevertheless defied solution. By discussing three such problems it is hoped to make the student aware of the "unfinished" business of mathematics and to stimulate their curiosity about our number system.

ACTIVITY 1: **The "3x + 1" problem.**

Pose the following question to the class:

Suppose you were a contestant on a game show and won. The host gives you a choice of two prizes — either take a flat $100, or he will compute your winnings based on the following formula:

1) Pick a number between 1 and 100.
2) If it's an odd number, triple it and add 1.
3) If it's an even number, divide it by 2.
4) Take the result you get and continue to apply steps (2) and (3).

The question is, if you decide to compute your winnings using the formula, will you be better or worse off than if you had chosen to accept the flat $100? In other words, where will repeating steps (2) and (3) lead you? Consider the following possible outcomes:

a. Because we multiply by 3 but only divide by 2, the numbers will slowly increase without bound and eventually you will bankrupt the network.
b. If, during iterations, we happen to bump into a power of 2 [what's the likelihood of this?] what will happen? Answer: We quickly wind up at 1.
c. If, during iterations, we happen to bump into a number we've hit before, what will happen? Answer: We wind up in an endless loop.

Which of these outcomes really occurs? To help answer this question, pick a few simple numbers and test [Note: choose 3, 8, 9, and 27. 3, 8, and 9 all end up in a 4-2-1 loop. Twenty-seven appears to grow without bound, but it, too, eventually ends in a 4-2-1 loop — after 111 steps!]. Now have the students pick relatively small numbers of their own and test the formula. In every case the number they choose will end in a 4-2-1 loop. The question is, Does this hold true for *every* number, no matter how large? Answer: No one knows! Although the pattern would tend to indicate all numbers end in a 4-2-1 loop and this has in fact been proven for all numbers up to 7×10^{11}, no one has been able to *prove* that this always happens.

ACTIVITY 2: **Goldbach's Conjecture**

In 1742 the Russian mathematician Christian Goldbach theorized that every even number (integer) greater than 2 could be expressed as the sum of two primes.

Procedure:

Starting with 4, this can easily be demonstrated for the first dozen or so even numbers. Have each student pick a fairly large even number (i.e., one greater than 100) and test to see whether Goldbach's Conjecture holds.

Goldbach's Conjecture is such a simple proposition and yet no one has been able to prove that it is (or isn't) true.

Corollary: Every odd number (integer) greater than 5 can be expressed as the sum of 3 primes.

Although the solution to these two problems are relatively insignificant in themselves, the *methods* derived to solve them may have considerable impact on the solution of more important and practical problems within and outside of pure mathematics. This may not be the case with the last problem, however, in that its solution would be significant in its own right as well as having potentially the most far reaching of consequences.

ACTIVITY 3: The last unsolved problem to be discussed is a topic that is practically taboo in mathematic circles: the prohibition of division by zero. The reason we "can't" divide by zero is that division by zero does not produce an answer consistent with the rules of our framework of mathematics.

Rather than encouraging avoidance of a problem because it doesn't apparently fit in with our rules, shouldn't we instead be encouraging students to consider this problem with the eventual goal of perhaps either finding the solution within the rules of our framework, or else discovering *new* rules that give meaning to division by zero?

Granted that this is manifestly a philosophical question. And yet it has importance not because we can *expect* a solution to the problems presented here, but because the very act of exploring these problems by students can be worthwhile in itself. If a knowledge of some of the classic unsolved problems in mathematics peaks the curiosity of a student and promotes thinking of mathematics and exploration of ideas, then we make academic and intellectual progress. And who is to say that such stimulation and motivation will not eventually lead to a solution of these problems and perhaps even to a resolution of the inadequacies in our present framework of mathematics?

EXTENDED LEARNING: Activities 1 and 2 are ideally suited for investigation using computers. Have students write a computer program to examine these problems listing the outcome of each trial. Note: More sophisticated programs will avoid redundant testing by avoiding numbers previously tested.

REFERENCES:

Gardner, Martin. *Wheels, Life and Other Mathematical Amusements*. New York: W. H. Freeman, 1983.

National Council of Teachers of Mathematics. *Enrichment Mathematics for High School*. 28th Yearbook. Reston, Virginia: NCTM, 1967.

WHAT'S IN A NUMBER?

GRADE LEVEL/ COURSE: 3

MATERIALS: Overhead transparency

INTRODUCTION/ OBJECTIVE: Students look at sets of numbers in a single focus (e.g., multiples of 5, even numbers, etc.). To help extend their sense of number, the students will name numbers with two or more characteristics.

PROCEDURE: Use this activity with small groups or individual students. Give each student a 5×8 card with a set of numbers on it (see samples). Explain that they are to select a number or numbers from the card which will fit the description you are going to read to them. (The numbers are placed at random).

ACTIVITY: Instruct the students to find and record a number from the card or chart which fits the sentence you are going to read. Only one number is necessary even though more than one may fit.

Step 1. Find a number which is both a:

— Multiple of 10 *and* a multiple of 3

— Multiple of 5 which is less than 30

— Multiple of 4 and a multiple of 3 but less than 15

Step 2. Find a number which is:

— One more than 2 × 7

— Two less than 4 × 3

— The same as 2 × 7 + 7

— Both a multiple of 5 and an even number

— Between 29 and 33

— Between 31 and 35 and is an odd number

EXTENDED LEARNING: Give students a card and have them write descriptions of the numbers using two or more characteristics.

SAMPLE CARD:

33 26 7
25
21 8
30 10 13
35
15
24
12 31

WHAT'S MISSING (PATTERNS)

GRADE LEVEL/ COURSE: 3–4

MATERIALS: Large patterns prepared by teacher for display on bulletin board. (See samples).

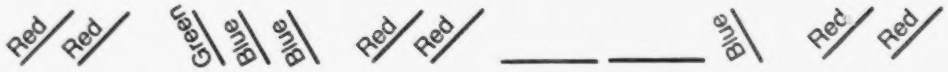
INTRODUCTION/ OBJECTIVE: Generalizing from patterns and classifications is basic to mathematics. Forming patterns may be the best or even the only technique for clarifying information. The student will classify objects by correctly completing a pattern.

PROCEDURE: This activity is used at intervals throughout the year. A different pattern can be displayed each week. After two or three days, students can sketch the figure which they think will complete the pattern and be ready to justify their decision by explaining how their sketch fits. In many cases different answers are possible.

ACTIVITY: (See examples).

EXTENDED LEARNING: Students will create their own patterns for other students to complete.

SAMPLE PATTERNS

1


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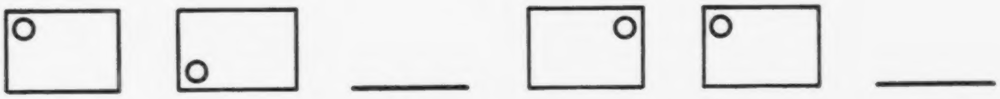
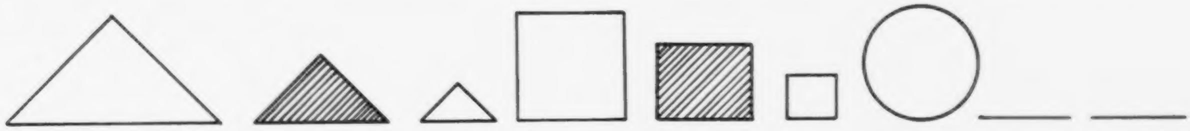


3


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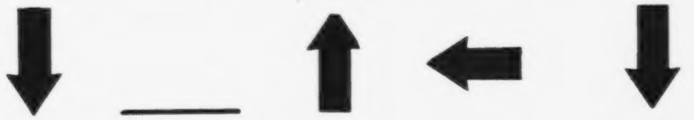

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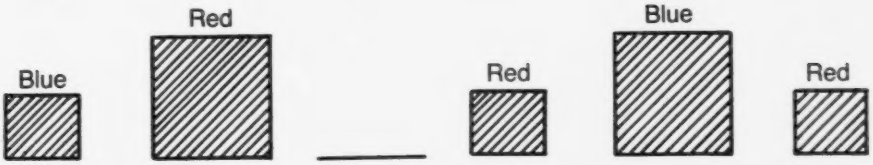

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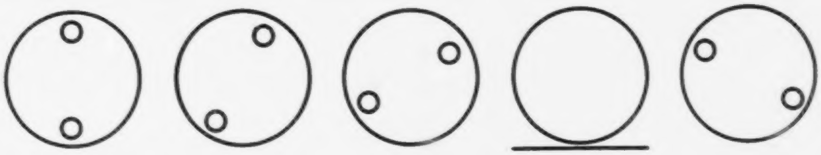

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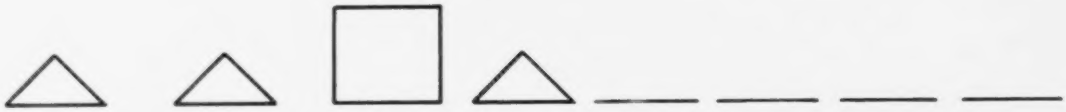

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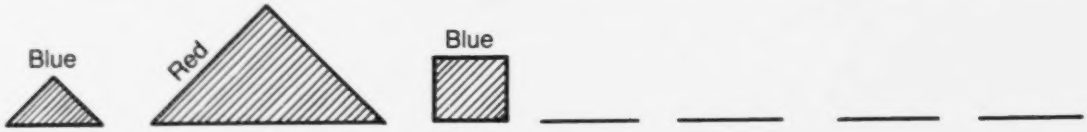

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